

# Values at non-positive integers of generalized Euler-Zagier multiple zeta-functions\*

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## Abstract.

We give new closed and explicit formulas for special values at non-positive integer points of generalized Euler-Zagier multiple zeta-functions and its partially twisted analogues. In the non-twisted case, we first prove these formulas for a small convenient class of these multiple zeta-functions and then use the analyticity of the values on the parameters defining the multiple zeta-functions to deduce the formulas in the general case. Also, for our aim we prove an extension of "Raabe's lemma" due to E. Friedman and A. Pereira (Lemma 2.4 of [9]). In the partially twisted case, we use a different method based on M. de Crisenoy's result [4] on the fully twisted case and the Mellin-Barnes integral formula.

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**Key words:** Multiple zeta-function, Euler-Zagier multiple zeta-functions, special values, meromorphic continuation, Bernoulli numbers.

## 1 Introduction and the statement of main results

Let  $\mathbb{N}$ ,  $\mathbb{N}_0$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  be the sets of positive integers, non-negative integers, rational integers, real numbers, and complex numbers, respectively.

Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  be two vectors of complex parameters such that  $\Re(\gamma_j) > 0$  and  $\Re(b_j) > -\Re(\gamma_1)$  for all  $j = 1, \dots, n$ .

The generalized Euler-Zagier multiple zeta-function is defined formally for  $n$ -tuples of complex variables  $\mathbf{s} = (s_1, \dots, s_n)$  by

$$\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_1 m_1 + \dots + \gamma_j m_j + b_j)^{s_j}}. \quad (1)$$

If  $b_1 = 0$ ,  $b_j = \gamma_2 + \dots + \gamma_j$  for all  $j = 2, \dots, n$  and  $\gamma_j = 1$  for all  $j = 1, \dots, n$ , then

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$\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b})$  coincides with the classical Euler-Zagier multiple zeta-function

$$\sum_{1 \leq m_1 < m_2 < \dots < m_n} \frac{1}{m_1^{s_1} \dots m_n^{s_n}}.$$

The generalized Euler-Zagier multiple zeta-function  $\zeta_n(\mathbf{s}; \boldsymbol{\gamma}; \mathbf{b})$  converges absolutely in the domain

$$\mathcal{D}_n := \{\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid \Re(s_j + \dots + s_n) > n + 1 - j \text{ for all } j = 1, \dots, n\} \quad (2)$$

(see [14]), and has a meromorphic continuation to the whole complex space  $\mathbb{C}^n$  whose poles are located in the union of the hyperplanes

$$s_j + \dots + s_n = (n + 1 - j) - k_j \quad (1 \leq j \leq n, \ k_1, \dots, k_n \in \mathbb{N}_0).$$

Moreover it is known that for  $n \geq 2$ , almost all non-positive integer points lie on the singular locus above and are points of indeterminacy. The evaluation of (limit) values of multiple zeta-functions at those points was first considered by S. Akiyama, S. Egami and Y. Tanigawa [1], and then studied further by [2], [18], [19], [13], [17], and [16].

In [13], Y. Komori proved that for any  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$  and  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$  such that  $\theta_j + \dots + \theta_n \neq 0$  for all  $j = 1, \dots, n$ , the limit

$$\zeta_n^\theta(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \rightarrow 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b}) \quad (3)$$

exists, and expressed this limit in terms of  $\mathbf{N}$ ,  $\boldsymbol{\theta}$  and generalized multiple Bernoulli numbers defined implicitly as coefficients of some multiple series.

Our first main result (i.e. Theorem 1) gives a closed explicit formula for  $\zeta_n^\theta(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b})$  in terms of  $\mathbf{N}$ ,  $\boldsymbol{\theta}$  and *only classical Bernoulli numbers*  $B_k$  ( $k \in \mathbb{N}_0$ ) defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}. \quad (4)$$

Before giving our result let us introduce a few notations:

1. For any vector  $\mathbf{x} = (x_1, \dots, x_n)$ , we write  $|\mathbf{x}| = x_1 + \dots + x_n$ ;
2. For  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{k} = (k_1, \dots, k_n)$ , we write

$$\mathbf{x}^{\mathbf{k}} = \prod_{i=1}^n x_i^{k_i}, \quad \binom{\mathbf{x}}{\mathbf{k}} = \prod_{i=1}^n \binom{x_i}{k_i};$$

3. For  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$  and  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ , we define

$$K(\mathbf{N}, \boldsymbol{\alpha}) := \left\{ j \in \{1, \dots, n\} \left| \sum_{i=j}^n N_i + (n+1-j) = \sum_{i=j}^n \alpha_i \right. \right\} \quad (5)$$

and

$$L(\mathbf{N}, \boldsymbol{\alpha}) := \{j \in \{1, \dots, n\} \mid \alpha_j \geq N_j + 1\}; \quad (6)$$

4. For  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$  and  $I \subset \{1, \dots, n\}$ , we define

$$\mathcal{J}(I, N) := \{\boldsymbol{\alpha} \in \mathbb{N}_0^n \mid K(\mathbf{N}, \boldsymbol{\alpha}) = I \text{ and } |L(\mathbf{N}, \boldsymbol{\alpha})| = |I|\}; \quad (7)$$

**Remark:**  $\mathcal{J}(I, N)$  is a finite set and  $\mathcal{J}(I, N) \subset \{0, \dots, |\mathbf{N}| + n\}^n$ . (See Lemma 2 for a proof of this fact).

5. For  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  we define the polynomial (in  $\mathbf{b}$ )  $c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k})$  (where  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n, |\mathbf{k}| \leq |\boldsymbol{\alpha}|$ ) as the coefficients of the polynomial  $\prod_{j=1}^n (\sum_{i=1}^j X_i + b_j)^{\alpha_j}$ ; that is

$$\prod_{j=1}^n \left( \sum_{i=1}^j X_i + b_j \right)^{\alpha_j} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n, \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k}) \mathbf{X}^{\mathbf{k}} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n, \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k}) X_1^{k_1} \dots X_n^{k_n}. \quad (8)$$

With these notations our first main result is the following:

**Theorem 1.** Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  be two vectors of complex parameters such that  $\Re(\gamma_j) > 0$  and  $\Re(b_j) > -\Re(\gamma_1)$  for all  $j = 1, \dots, n$ . Let  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$  such that  $\theta_j + \dots + \theta_n \neq 0$  for all  $j = 1, \dots, n$ . Then, for any  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ , the limit

$$\zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) := \lim_{t \rightarrow 0} \zeta_n(-\mathbf{N} + t\boldsymbol{\theta}; \boldsymbol{\gamma}; \mathbf{b})$$

exists, and is explicitly given by

$$\begin{aligned} & \zeta_n^{\boldsymbol{\theta}}(-\mathbf{N}; \boldsymbol{\gamma}; \mathbf{b}) \\ = & \sum_{I \subset \{1, \dots, n\}} \sum_{\boldsymbol{\alpha} \in \mathcal{J}(I, N)} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n, \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} \frac{c_n(\mathbf{b}; \boldsymbol{\alpha}, \mathbf{k}) (-1)^{n-|I|+\sum_{j \in L(\mathbf{N}, \boldsymbol{\alpha})} (\alpha_j - N_j)} \prod_{j \notin L(\mathbf{N}, \boldsymbol{\alpha})} \binom{N_j}{\alpha_j}}{\prod_{j \in L(\mathbf{N}, \boldsymbol{\alpha})} \alpha_j \binom{\alpha_j - 1}{N_j} \prod_{j \notin I} \left( \sum_{i=j}^n N_i + (n+1-j) - \sum_{i=j}^n \alpha_i \right)} \\ & \times \left( \gamma_1^{|\mathbf{N}| - |\boldsymbol{\alpha}| + n + k_1 - 1} \prod_{j=2}^n \gamma_j^{k_j - 1} \right) \left( \frac{\prod_{j \in I(\boldsymbol{\alpha}, N)} \theta_j}{\prod_{j \in I} (\theta_j + \dots + \theta_n)} \right) \left( \prod_{j=1}^n B_{k_j} \right). \quad (9) \end{aligned}$$

We will prove this theorem in Section 2. An essential idea in the proof is to prove these formulas first for a small convenient class of these multiple zeta-functions and then use the analyticity of the values on the parameters defining the multiple zeta-functions to deduce the formulas in the general case. We also prove an extension of a lemma of "Raabe type" due to E. Friedman and A. Pereira (Lemma 2.4 of [9]) and use it in the proof.

Next we consider the twisted multiple zeta-functions. Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  be two vectors of complex parameters such that  $\Re(\gamma_j) > 0$  and  $\Re(b_j) > -\Re(\gamma_1)$  for all  $j = 1, \dots, n$ .

Let  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ , and let  $\boldsymbol{\mu}_k = (\mu_1, \dots, \mu_k) \in (\mathbb{T} \setminus \{1\})^k$ , where  $k \in \{0, \dots, n\}$ . The partially twisted generalized Euler-Zagier multiple zeta-function is defined formally for  $n$ -tuples of complex variables  $\mathbf{s} = (s_1, \dots, s_n)$  by

$$\zeta_{n,k}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k) = \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{\prod_{j=1}^k \mu_j^{m_j}}{\prod_{j=1}^n (\gamma_1 m_1 + \dots + \gamma_j m_j + b_j)^{s_j}} \quad (10)$$

(when  $k = 0$ , we understand that the numerator on the right-hand side is 1) which is absolutely convergent in the domain  $\mathcal{D}_n$  (see (2)). The meromorphic continuation and the location of singularities of the function  $\zeta_{n,k}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k)$  are discussed in [12] (which are partly announced in [11]).

When  $k = n$ , M. de Crisenoy [4] proved that  $\zeta_{n,n}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu})$  (where  $\boldsymbol{\mu} = \boldsymbol{\mu}_n$ ) are entire in  $\mathbf{s}$  and obtained an explicit formula for the values of  $\zeta_{n,n}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu})$  at non-positive integer points in terms of Lerch zeta-functions.

If  $k < n$ ,  $\zeta_{n,k}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k)$  has meromorphic continuation to  $\mathbb{C}^n$ . Moreover, the complexity of its set of singularities and therefore the complexity of its special values, increases when  $k$  decreases. In our Theorem 1 above we handle the case  $k = 0$  by a method different from that in [4]. Our following two main theorems (i.e. Theorem 2 and Theorem 3, proved in Section 3 and Section 4, respectively) deal with the cases  $k = n - 1$  and  $k = n - 2$ . In these cases we use in addition to de Crisenoy's result [4], the Mellin-Barnes formula to determine the set of singularities and the values of  $\zeta_{n,k}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k)$  at non-positive integers.

We prepare some more notations.

1. For any  $\mu \in \mathbb{T}$ , let  $\zeta_\mu(s) = \sum_{m=1}^{\infty} \mu^m m^{-s}$  be the Lerch zeta-function;
2. For any  $a \in \mathbb{C} \setminus (-\mathbb{N}_0)$ , let  $\zeta(s, a) = \sum_{m=0}^{\infty} (m+a)^{-s}$  be the Hurwitz zeta-function (as for the definition of  $\zeta(s, a)$  for any  $a \in \mathbb{C} \setminus (-\mathbb{N}_0)$ , see [15, Lemma 1]);

3. For  $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ , and  $k < n$ , denote  $\mathbf{s}_k = (s_1, \dots, s_k)$ . Similarly we use the notation  $\boldsymbol{\gamma}_k, \mathbf{b}_k, \boldsymbol{\mu}_k$  etc.

4. For any  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$  and any  $l \in \mathbb{Z}$ , let

$$\mathbf{N}_{n-1}^*(l) = (N_1, \dots, N_{n-2}, N_{n-1} + N_n + l).$$

**Theorem 2.** Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  be two vectors of complex parameters such that  $\Re(\gamma_j) > 0$ ,  $\Re(b_j) > -\Re(\gamma_1)$  and  $(b_n - b_{n-1})/\gamma_n \notin -\mathbb{N}_0$  for all  $j = 1, \dots, n$ . Let  $\boldsymbol{\mu}_{n-1} = (\mu_1, \dots, \mu_{n-1}) \in (\mathbb{T} \setminus \{1\})^{n-1}$ . Then,  $\zeta_{n,n-1}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-1})$  has meromorphic continuation to the whole space  $\mathbb{C}^n$  and its poles are located only in the hyperplane  $s_n = 1$ . Furthermore, for any  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ ,

$$\begin{aligned} & \zeta_{n,n-1}(-\mathbf{N}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-1}) \\ &= -\frac{1}{N_n + 1} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{n-1} \\ |\mathbf{k}| \leq |\mathbf{N}_{n-1}^*(1)|}} c_{n-1}(\mathbf{b}'_{n-1}; \mathbf{N}_{n-1}^*(1), \mathbf{k}) \gamma_n^{-1} \prod_{j=1}^{n-1} \gamma_j^{k_j} \zeta_{\mu_j}(-k_j) \\ &+ \sum_{l=0}^{N_n} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{n-1} \\ |\mathbf{k}| \leq |\mathbf{N}_{n-1}^*(-l)|}} c_{n-1}(\mathbf{b}'_{n-1}; \mathbf{N}_{n-1}^*(-l), \mathbf{k}) \gamma_n^l \left( \prod_{j=1}^{n-1} \gamma_j^{k_j} \zeta_{\mu_j}(-k_j) \right) \zeta \left( -l; \frac{b_n - b_{n-1}}{\gamma_n} \right), \end{aligned} \tag{11}$$

where  $\mathbf{b}'_{n-1} = (b_1, b'_2, \dots, b'_{n-1})$  with  $b'_j = b_j - (\gamma_2 + \dots + \gamma_j)$  ( $2 \leq j \leq n-1$ ) and  $c_{n-1}(\mathbf{b}'_{n-1}; \mathbf{N}_{n-1}^*(-l), \mathbf{k})$  is defined as in (8).

**Theorem 3.** Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  be two vectors of complex parameters such that  $\Re(\gamma_j) > 0$ ,  $\Re(b_j) > -\Re(\gamma_1)$  and  $(b_n - b_{n-1})/\gamma_n \notin -\mathbb{N}_0$  for all  $j = 1, \dots, n$ . Let  $\boldsymbol{\mu}_{n-2} = (\mu_1, \dots, \mu_{n-2}) \in (\mathbb{T} \setminus \{1\})^{n-2}$ . Then,  $\zeta_{n,n-2}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-2})$  has meromorphic continuation to the whole space  $\mathbb{C}^n$  and its singularities are located in the hyperplanes

$$s_n = 1 \quad \text{and} \quad s_{n-1} + s_n = k \quad (k \in \mathbb{Z}, k \leq 2).$$

Furthermore, for any  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ , we have as  $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$  tends to  $(0, \dots, 0)$ :

$$\begin{aligned} & \zeta_{n,n-2}(-\mathbf{N} + \boldsymbol{\delta}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-2}) \\ &= -\frac{1}{N_n + 1} \zeta_{n-1,n-2}(-\mathbf{N}_{n-1}^*(-1), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \gamma_n^{-1} \\ &+ \sum_{l=0}^{N_n} \binom{N_n}{l} \zeta_{n-1,n-2}(-\mathbf{N}_{n-1}^*(l), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \zeta \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l \end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^{N_{n-1}+1} N_n! N_{n-1}!}{(N_{n-1} + N_n + 1)!} \left( \frac{\delta_n + O(\delta_n^2)}{\delta_{n-1} + \delta_n} \right) \\
& \times \zeta_{n-2, n-2}(-\mathbf{N}_{n-2} + \boldsymbol{\delta}_{n-2}, \boldsymbol{\gamma}_{n-2}, \mathbf{b}_{n-2}, \boldsymbol{\mu}_{n-2}) \\
& \times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_{n-1}^{-1} \gamma_n^{N_{n-1} + N_n + 1} \\
& + O \left( \max_{1 \leq j \leq n} |\delta_j| \right).
\end{aligned}$$

As a corollary, we obtain the following result:

**Corollary 1.** *Assume that the assumptions of Theorem 3 hold. Let  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$  and  $\theta \in \mathbb{C}$ . Then, the limit*

$$\zeta_{n, n-2}^\theta(-\mathbf{N}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-2}) := \lim_{\boldsymbol{\delta} \rightarrow \mathbf{0}, \frac{\delta_n}{\delta_{n-1} + \delta_n} \rightarrow \theta} \zeta_{n, n-2}(-\mathbf{N} + \boldsymbol{\delta}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-2})$$

exists and is given by

$$\begin{aligned}
& \zeta_{n, n-2}^\theta(-\mathbf{N}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-2}) \\
& = -\frac{1}{N_n + 1} \zeta_{n-1, n-2}(-\mathbf{N}_{n-1}^*(-1), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \gamma_n^{-1} \\
& + \sum_{l=0}^{N_n} \binom{N_n}{l} \zeta_{n-1, n-2}(-\mathbf{N}_{n-1}^*(l), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \zeta \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l \\
& + \frac{(-1)^{N_{n-1}+1} N_n! N_{n-1}!}{(N_{n-1} + N_n + 1)!} \zeta_{n-2, n-2}(-\mathbf{N}_{n-2}, \boldsymbol{\gamma}_{n-2}, \mathbf{b}_{n-2}, \boldsymbol{\mu}_{n-2}) \theta \\
& \times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_{n-1}^{-1} \gamma_n^{N_{n-1} + N_n + 1}.
\end{aligned}$$

When  $n = 2$ , Theorem 3 gives the result on the (non-twisted) double zeta-function, which coincides with [16, Corollary 5.2].

The argument to prove Theorem 3 can be extended to the case  $k \leq n - 3$ , to obtain the same type of explicit formulas. However, for smaller values of  $k$ , more and more relevant singularities will appear, so the description of indeterminacy will be much more complicated.

Let  $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{N}^n$ , and consider

$$\zeta_{n, k}(\mathbf{s}, \mathbf{h}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k) = \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{\prod_{j=1}^k \mu_j^{m_j}}{\prod_{j=1}^n (\gamma_1 m_1^{h_1} + \dots + \gamma_j m_j^{h_j} + b_j)^{s_j}}.$$

This is obviously a generalization of  $\zeta_{n,k}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k)$ , and can be treated (see Section 5) in a quite similar way as in the linear case in Theorems 2 and 3.

More generally, we can consider the multiple Dirichlet series of the form

$$Z_{n,k}(\mathbf{s}, \mathbf{P}, \boldsymbol{\mu}_k) = \sum_{m_1, \dots, m_n \geq 1} \frac{\prod_{j=1}^k \mu_j^{m_j}}{\prod_{j=1}^n P_j(m_1, \dots, m_j)^{s_j}}, \quad (12)$$

where the  $P_j$  are polynomials. It follows from the method of [5] (see also [6]) that these series have meromorphic continuation to  $\mathbb{C}^n$  for fairly general class of polynomials  $P_j$ . However, the set of singularities is not easy to determine in this general setting. In our forthcoming work [7], we will overcome this problem and obtain some partial extensions of our results in the present paper to  $Z_{n,k}(\mathbf{s}, \mathbf{P}, \boldsymbol{\mu}_k)$ .

## 2 Proof of Theorem 1

### 2.1 Some useful lemmas

**Lemma 1.** *Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  such that  $\Re(\gamma_j) > 0$  for any  $j = 1, \dots, n$ . Define for any  $\mathbf{s} = (s_1, \dots, s_n) \in \mathcal{D}_n$*

$$Y_n(\mathbf{s}; \boldsymbol{\gamma}) := \int_{(1, \infty) \times (0, \infty)^{n-1}} \prod_{j=1}^n \left( \sum_{i=1}^j \gamma_i x_i \right)^{-s_j} dx_n \dots dx_1. \quad (13)$$

*Then, for any  $\mathbf{s} = (s_1, \dots, s_n) \in \mathcal{D}_n$ ,*

$$Y_n(\mathbf{s}; \boldsymbol{\gamma}) = \frac{\gamma_1^{-s_1 - \dots - s_n + n}}{(\gamma_1 \dots \gamma_n) \prod_{j=1}^n (s_j + \dots + s_n + j - n - 1)}.$$

*In particular,  $Y_n(\mathbf{s}; \boldsymbol{\gamma})$  has a meromorphic continuation to the whole complex space  $\mathbb{C}^n$  and its polar locus is the set*

$$\bigcup_{j=1}^n \{ \mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid s_j + \dots + s_n = n + 1 - j \}.$$

**Proof of Lemma 1:** Just integrate first with respect to the variable  $x_n$  and then with respect to  $x_{n-1}$  etc.  $\square$

**Lemma 2.** *Let  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$  and  $I \subset \{1, \dots, n\}$ . The set  $J(I, N)$  defined by (7) is a finite set and  $\mathcal{J}(I, N) \subset \{0, \dots, |\mathbf{N}| + n\}^n$ .*

**Proof of Lemma 2:**

Denote by  $j_1, \dots, j_q$  the elements of the set  $I$ , where  $q = |I|$ . We assume without loss of generality that  $j_1 < j_2 < \dots < j_q$ .

Let  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{J}(I, N)$ . It follows that for any  $k = 2, \dots, q$ ,

$$\begin{aligned}
\sum_{j=j_{k-1}}^{j_k-1} \alpha_j &= \sum_{j=j_{k-1}}^n \alpha_j - \sum_{j=j_k}^n \alpha_j \\
&= \sum_{j=j_{k-1}}^n N_j + (n+1-j_{k-1}) - \sum_{j=j_k}^n N_j - (n+1-j_k) \\
&= \sum_{j=j_{k-1}}^{j_k-1} N_j + (j_k - j_{k-1}) \\
&\geq \sum_{j=j_{k-1}}^{j_k-1} N_j + 1,
\end{aligned}$$

hence  $[j_{k-1}, j_k) \cap L(\mathbf{N}, \alpha) \neq \emptyset$  for any  $k = 2, \dots, q$ . Moreover, the identity

$$\sum_{j=j_q}^n \alpha_j = \sum_{j=j_q}^n N_j + (n+1-j_q) \geq \sum_{j=j_q}^n N_j + 1$$

implies also that  $[j_q, n] \cap L(\mathbf{N}, \alpha) \neq \emptyset$ . Since  $|L(\mathbf{N}, \alpha)| = q$ , the above observation implies that  $\min L(\mathbf{N}, \alpha) \geq j_1$ . We deduce that for any  $j \in L(\mathbf{N}, \alpha)$ ,

$$\alpha_j \leq \sum_{j=j_1}^n \alpha_j = \sum_{j=j_1}^n N_j + (n+1-j_1) \leq |\mathbf{N}| + n.$$

If  $j \notin L(\mathbf{N}, \alpha)$ , obviously  $\alpha_j < N_j + 1 \leq |\mathbf{N}| + n$ . This ends the proof of Lemma 2.  $\square$

The following lemma is crucial for our proof of Theorem 1.

**Lemma 3.** Let  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$  and  $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$  such that  $\theta_j + \dots + \theta_n \neq 0$  for all  $j = 1, \dots, n$ . Set

$$\delta := \frac{1}{2} \min \{ (1 + |\theta_j|)^{-1}, |\theta_j + \dots + \theta_n|^{-1}; j = 1, \dots, n \} \in (0, 1/2).$$

Let  $U_\delta := \{t \in \mathbb{C} \mid |t| < \delta\}$ , and for  $t \in U_\delta \setminus \{0\}$  define

$$G_{\mathbf{N}, \alpha, \theta}(t) := \frac{\prod_{j=1}^n \binom{N_j - t\theta_j}{\alpha_j}}{\prod_{j=1}^n \left( t - \frac{(N_j + \dots + N_n) + (n+1-j) - (\alpha_j + \dots + \alpha_n)}{\theta_j + \dots + \theta_n} \right)}. \quad (14)$$

Let  $q = q(\mathbf{N}, \alpha) := |K(\mathbf{N}, \alpha)|$  and  $q' = q'(\mathbf{N}, \alpha) := |L(\mathbf{N}, \alpha)|$ , where  $K(\mathbf{N}, \alpha)$ ,  $L(\mathbf{N}, \alpha)$  are defined by (5), (6), respectively. Then,



1.  $q' \geq q$ ;

2.  $G_{\mathbf{N}, \alpha, \theta}(t)$  is analytic in the disk  $U_\delta$  and there exists a constant  $C = C(\mathbf{N}, \theta) > 0$  (which is independent of  $\alpha$ ) such that

$$|G_{\mathbf{N}, \alpha, \theta}(t)| \leq C |t|^{q'-q} \quad \text{for all } t \in U_\delta.$$

3. If  $q' > q$ , then  $G_{\mathbf{N}, \alpha, \theta}(0) = 0$ ;

4. If  $q' = q$ , then

(a)

$$G_{\mathbf{N}, \alpha, \theta}(0) = \frac{(-1)^{n-q} \left( \prod_{j \in L(\mathbf{N}, \alpha)} \frac{(-1)^{\alpha_j - N_j} \theta_j}{\alpha_j \binom{\alpha_j - 1}{N_j}} \right) \prod_{j \notin L(\mathbf{N}, \alpha)} \binom{N_j}{\alpha_j}}{\prod_{j \notin K(\mathbf{N}, \alpha)} \left( \frac{(N_j + \dots + N_n) + (n+1-j) - (\alpha_j + \dots + \alpha_n)}{\theta_j + \dots + \theta_n} \right)};$$

(b)  $\alpha \in \mathcal{J}(K(\mathbf{N}, \alpha), N) \subset \{0, \dots, |\mathbf{N}| + n\}^n$  (see Lemma 2).

### Proof of Lemma 3:

• **Proof of point 1:** Repeat the argument of the proof of the previous lemma with  $I = K(\mathbf{N}, \alpha)$ . It follows that  $q' = |L(\mathbf{N}, \alpha)| \geq q$ .  $\square$

• **Proof of point 2:** First it is easy to see that  $G_{\mathbf{N}, \alpha, \theta}(t)$  is *analytic* in all the pointed disk  $U_\delta \setminus \{0\}$ . (Since  $(N_j + \dots + N_n) + (n+1-j) - (\alpha_j + \dots + \alpha_n) \in \mathbb{Z}$ , if it is not zero, then  $\frac{(N_j + \dots + N_n) + (n+1-j) - (\alpha_j + \dots + \alpha_n)}{\theta_j + \dots + \theta_n} \geq 2\delta$ .) Moreover we have for any  $j = 1, \dots, n$  and any  $t \in U_\delta$ ,

$$\binom{N_j - t\theta_j}{\alpha_j} = \frac{1}{\alpha_j!} \prod_{k=0}^{\alpha_j-1} (N_j - t\theta_j - k). \quad (15)$$

It follows that

1. If  $\alpha_j \leq N_j$ , then  $\binom{N_j - t\theta_j}{\alpha_j} \big|_{t=0} = \binom{N_j}{\alpha_j}$  and for any  $t \in U_\delta$ :

$$\left| \binom{N_j - t\theta_j}{\alpha_j} \right| \leq \frac{1}{\alpha_j!} \prod_{k=0}^{\alpha_j-1} (N_j + 1 - k) \leq (N_j + 1)!;$$

2. If  $\alpha_j \geq N_j + 1$ , then  $\binom{N_j - t\theta_j}{\alpha_j} \big|_{t=0} = \binom{N_j}{\alpha_j} = 0$  and for any  $t \in U_\delta$ :

$$\begin{aligned} \left| \binom{N_j - t\theta_j}{\alpha_j} \right| &\leq \frac{|t\theta_j|}{\alpha_j!} \prod_{k=0}^{N_j-1} (N_j - k + 1) \prod_{k=N_j+1}^{\alpha_j-1} (k - N_j + 1) \\ &= |t\theta_j| \frac{N_j!}{\alpha_j(\alpha_j - 1) \cdots (\alpha_j - N_j + 1)} (N_j + 1) \leq (N_j + 1) |t\theta_j|. \end{aligned}$$

We deduce that for any  $t \in U_\delta \setminus \{0\}$ ,

$$\begin{aligned}
& |G_{\mathbf{N}, \alpha, \theta}(t)| \\
& \ll_{\mathbf{N}, \theta} \frac{\prod_{j \in L(\mathbf{N}, \alpha)} |t|}{\prod_{j \in K(\mathbf{N}, \alpha)} |t| \prod_{j \notin K(\mathbf{N}, \alpha)} \left| t - \frac{(N_j + \dots + N_n) + (n+1-j) - (\alpha_j + \dots + \alpha_n)}{\theta_j + \dots + \theta_n} \right|} \\
& \ll_{\mathbf{N}, \theta} \frac{|t|^{q'-q}}{\prod_{j \notin K(\mathbf{N}, \alpha)} (|t(\theta_j + \dots + \theta_n) - (N_j + \dots + N_n) - (n+1-j) + (\alpha_j + \dots + \alpha_n)|)} \\
& \ll_{\mathbf{N}, \theta} \frac{|t|^{q'-q}}{\prod_{j \notin K(\mathbf{N}, \alpha)} (1/2)} \ll_{\mathbf{N}, \theta} |t|^{q'-q}.
\end{aligned}$$

It follows that  $G_{\mathbf{N}, \alpha, \theta}(t)$  is analytic in the whole disk  $U_\delta$  and verifies in it the uniform estimate  $G_{\mathbf{N}, \alpha, \theta}(t) \ll_{\mathbf{N}, \theta} |t|^{q'-q}$ .  $\square$

• **Proof of point 3:** Follows from point 2.  $\square$

• **Proof of point 4:** The identity (15) imply that if  $\alpha_j \geq N_j + 1$ , then

$$\binom{N_j - t\theta_j}{\alpha_j} \sim_{t \rightarrow 0} \frac{-t\theta_j}{\alpha_j!} \prod_{k=0}^{N_j-1} (N_j - k) \prod_{k=N_j+1}^{\alpha_j-1} (N_j - k) \sim_{t \rightarrow 0} \frac{t\theta_j (-1)^{\alpha_j - N_j}}{\alpha_j \binom{\alpha_j - 1}{N_j}}.$$

It follows that

$$G_{\mathbf{N}, \alpha, \theta}(t) \sim_{t \rightarrow 0} \frac{(-1)^{n-q} \left( \prod_{j \in L(\mathbf{N}, \alpha)} \frac{(-1)^{\alpha_j - N_j} \theta_j}{\alpha_j \binom{\alpha_j - 1}{N_j}} \right) \prod_{j \notin L(\mathbf{N}, \alpha)} \binom{N_j}{\alpha_j}}{\prod_{j \notin K(\mathbf{N}, \alpha)} \left( \frac{(N_j + \dots + N_n) + (n+1-j) - (\alpha_j + \dots + \alpha_n)}{\theta_j + \dots + \theta_n} \right)}.$$

This ends the proof of point 4 and therefore ends the proof of Lemma 3.  $\square$

## 2.2 The key propositions

Now we introduce a class of multivariate zeta functions which are slightly more general than that considered in Theorem 1. We are working in this slightly more general class because it is more suitable for induction arguments.

Let  $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{N}^n$ . Set  $q = |\mathbf{q}| = q_1 + \dots + q_n$ . We will use the notation  $\mathbf{s} = (s_{1,1}, \dots, s_{1,q_1}, \dots, s_{j,1}, \dots, s_{j,q_j}, \dots, s_{n,1}, \dots, s_{n,q_n})$  for elements of  $\mathbb{C}^q$ , and denote  $|\mathbf{s}| = s_{1,1} + \dots + s_{1,q_1} + \dots + s_{j,1} + \dots + s_{j,q_j} + \dots + s_{n,1} + \dots + s_{n,q_n}$ . Let  $\varepsilon \geq 0$  (notice, here, we admit the case  $\varepsilon = 0$ ),  $\gamma \in \mathbb{C}^n$ , and define

$$\begin{aligned}
W_\varepsilon(\mathbf{q}, n) &:= \{(\mathbf{u}, \gamma) \in \mathbb{C}^q \times \mathbb{C}^n \mid \Re(\gamma_j) > \varepsilon \text{ and } \Re(u_{j,k} + \gamma_1) > \varepsilon \\
&\quad \text{for all } j = 1, \dots, n \text{ and } k = 1, \dots, q_j\}, \\
V_{\varepsilon, \mathbf{q}}(\gamma) &:= \{\mathbf{u} \in \mathbb{C}^q \mid \Re(u_{j,k} + \gamma_1) > \varepsilon \text{ for all } j = 1, \dots, n \text{ and } k = 1, \dots, q_j\},
\end{aligned}$$

and

$$\mathcal{D}_{n,\mathbf{q}} := \left\{ \mathbf{s} \in \mathbb{C}^q \mid \Re \left( \sum_{i=j}^n \sum_{k=1}^{q_i} s_{i,k} \right) > n+1-j \text{ for all } j=1, \dots, n \right\}.$$

For  $\mathbf{s} \in \mathcal{D}_{n,\mathbf{q}}$  and  $(\mathbf{u}, \gamma) \in W_0(\mathbf{q}, n)$ , define

$$Y_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma) := \int_{[1,\infty) \times [0,\infty)^{n-1}} \prod_{j=1}^n \prod_{k=1}^{q_j} (\gamma_1 x_1 + \dots + \gamma_j x_j + u_{j,k})^{-s_{j,k}} dx_n \dots dx_1 \quad (16)$$

and

$$Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n \prod_{k=1}^{q_j} (\gamma_1 m_1 + \dots + \gamma_j m_j + u_{j,k})^{s_{j,k}}}. \quad (17)$$

The multiple zeta-function  $Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$  is convergent absolutely in the region  $\mathcal{D}_{n,\mathbf{q}}$ , and in this region

$$Y_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma) = \int_{[0,1]^n} Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}_{\mathbf{q}}(\mathbf{b}); \gamma) d\mathbf{b}. \quad (18)$$

Here,  $\mathbf{u}_{\mathbf{q}}(\mathbf{b}) \in \mathbb{C}^q$  is given by

$$\mathbf{u}_{\mathbf{q}}(\mathbf{b}) = (u_{1,1}(\mathbf{b}), \dots, u_{1,q_1}(\mathbf{b}), \dots, u_{j,1}(\mathbf{b}), \dots, u_{j,q_j}(\mathbf{b}), \dots, u_{n,1}(\mathbf{b}), \dots, u_{n,q_n}(\mathbf{b})),$$

where  $\mathbf{b} = (b_1, \dots, b_n) \in [0, 1]^n$  and  $u_{j,k}(\mathbf{b}) = u_{j,k} + \sum_{i=1}^j \gamma_i b_i$  for all  $j = 1, \dots, n$  and all  $k = 1, \dots, q_j$ .

Now we state a proposition, which gives several analytic properties of  $Y_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$  and  $Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$ .

**Proposition 1.** *1. The functions  $\mathbf{s} \mapsto Y_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$  and  $\mathbf{s} \mapsto Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$  can be meromorphically continued to  $\mathbb{C}^q$  and their poles are located in the set*

$$\mathcal{P}_{n,\mathbf{q}} := \bigcup_{j=1}^n \bigcup_{k_j \in \mathbb{N}_0} \left\{ \mathbf{s} \in \mathbb{C}^q \mid \sum_{i=j}^n \sum_{k=1}^{q_i} s_{i,k} = n+1-j-k_j \right\}.$$

*Therefore (18) is valid for all  $\mathbf{s} \in \mathbb{C}^n \setminus \mathcal{P}_{n,\mathbf{q}}$ .*

*2. For any fixed  $\boldsymbol{\omega} \in \mathbb{C}^q$  and any  $\boldsymbol{\theta} \in \mathbb{C}^q$  such that  $\sum_{i=j}^n \sum_{k=1}^{q_i} \theta_{i,k} \neq 0$  for all  $j = 1, \dots, n$ , there exist  $\delta = \delta(\boldsymbol{\omega}, \boldsymbol{\theta}) > 0$  and  $M = M(\boldsymbol{\omega}, \boldsymbol{\theta}) > 0$  such that*

*(a)  $(t, \mathbf{u}, \gamma) \mapsto t^n Y_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta}; \mathbf{u}; \gamma)$  and  $(t, \mathbf{u}, \gamma) \mapsto t^n Z_{n,\mathbf{q}}(\boldsymbol{\omega} + t\boldsymbol{\theta}; \mathbf{u}; \gamma)$  are analytic in the domain  $U_\delta \times W_0(\mathbf{q}, n)$ ;*

(b) for any  $\varepsilon > 0$  and any  $\gamma \in \mathbb{C}^n$  such that  $\Re(\gamma_j) > \varepsilon$  for all  $j = 1, \dots, n$ , we have

$$|t^n Y_{n,\mathbf{q}}(\omega + t\theta; \mathbf{u}; \gamma)| \ll_{\omega, \theta, \gamma, \varepsilon} (1 + |\mathbf{u}|)^M$$

and

$$|t^n Z_{n,\mathbf{q}}(\omega + t\theta; \mathbf{u}; \gamma)| \ll_{\omega, \theta, \gamma, \varepsilon} (1 + |\mathbf{u}|)^M$$

uniformly in  $(t, \mathbf{u}) \in U_\delta \times V_{\varepsilon, \mathbf{q}}(\gamma)$ .

Proposition 1 implies the following key result:

**Corollary 2.** Let  $\omega \in \mathbb{C}^q$  and any  $\theta \in \mathbb{C}^q$  such that  $\sum_{i=j}^n \sum_{k=1}^{q_i} \theta_{i,k} \neq 0$  for all  $j = 1, \dots, n$ . For any  $(\mathbf{u}, \gamma) \in W_0(\mathbf{q}, n)$ , each of the meromorphic functions

$$t \mapsto Y_{n,\mathbf{q}}(\omega + t\theta; \mathbf{u}; \gamma) \text{ and } t \mapsto Z_{n,\mathbf{q}}(\omega + t\theta; \mathbf{u}; \gamma)$$

has at most a pole of order  $n$  at  $t = 0$ . Moreover if we write

$$Y_{n,\mathbf{q}}(\omega + t\theta; \mathbf{u}; \gamma) = \sum_{k=0}^n \frac{y_{-k,\mathbf{q}}(\mathbf{u}; \omega, \theta; \gamma)}{t^k} + O(t) \quad \text{as } t \rightarrow 0,$$

and

$$Z_{n,\mathbf{q}}(\omega + t\theta; \mathbf{u}; \gamma) = \sum_{k=0}^n \frac{z_{-k,\mathbf{q}}(\mathbf{u}; \omega, \theta; \gamma)}{t^k} + O(t) \quad \text{as } t \rightarrow 0.$$

Then, for any  $k = 0, \dots, n$ ,

1. the functions

$$(\mathbf{u}, \gamma) \mapsto y_{-k,\mathbf{q}}(\mathbf{u}; \omega, \theta; \gamma) \text{ and } (\mathbf{u}, \gamma) \mapsto z_{-k,\mathbf{q}}(\mathbf{u}; \omega, \theta; \gamma)$$

are analytic in the domain  $W_0(\mathbf{q}, n)$  and

$$y_{-k,\mathbf{q}}(\mathbf{u}; \omega, \theta; \gamma) = \int_{[0,1]^n} z_{-k,\mathbf{q}}(\mathbf{u}_{\mathbf{q}}(\mathbf{b}); \omega, \theta; \gamma) d\mathbf{b} \quad (19)$$

holds in that domain.

2. There exists an  $M = M(\omega, \theta) > 0$  such that for any  $\varepsilon > 0$ , and any  $\gamma \in \mathbb{C}^n$  such that  $\Re(\gamma_j) > \varepsilon$  for any  $j = 1, \dots, n$ , we have uniformly in  $\mathbf{u} \in V_{\varepsilon, \mathbf{q}}(\gamma)$ :

$$y_{-k,\mathbf{q}}(\mathbf{u}; \omega, \theta; \gamma) \ll_{\omega, \theta, \gamma, \varepsilon} (1 + |\mathbf{u}|)^M \text{ and } z_{-k,\mathbf{q}}(\mathbf{u}; \omega, \theta; \gamma) \ll_{\omega, \theta, \gamma, \varepsilon} (1 + |\mathbf{u}|)^M.$$

### Deduction of Corollary 2 from Proposition 1:

The corollary follows from point 2 of Proposition 1 by applying the Cauchy formula and the theorem of analyticity under the integral sign. The identity (19) follows by using in addition the equality (18).  $\square$

### Proof of Proposition 1:

The proof of the proposition for  $Y_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$  is similar (and more easier) than its proof for  $Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$ . So we will give here only the proof for  $Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$ . We will prove the proposition for  $Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$  by induction on  $n$ .

#### • Proof of Proposition 1 in the case $n = 1$ :

For any  $(\mathbf{u}, \gamma) = ((u_1, \dots, u_q), \gamma) \in W_0(q, 1)$  and any  $\mathbf{s} = (s_1, \dots, s_q) \in \mathcal{D}_{1,q}$ , we have

$$Z_{1,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma) = \sum_{m \geq 1} \frac{1}{\prod_{k=1}^q (\gamma m + u_k)^{s_k}}.$$

Let  $K \in \mathbb{N}_0$ , and for any  $m \geq 1$ , put  $\psi_m(z) = \prod_{k=1}^q \left(1 + \frac{u_k}{\gamma m} z\right)^{-s_k}$ . Since

$$\prod_{k=1}^q (\gamma m + u_k)^{-s_k} = \prod_{k=1}^q (\gamma m)^{-s_k} \psi_m(1),$$

applying Taylor's formula with remainder ([8, (3.4)]) to the function  $\psi_m(z)$ , we obtain that for any  $m \geq 1$ ,

$$\begin{aligned} \prod_{k=1}^q (\gamma m + u_k)^{-s_k} &= \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| \leq K}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \gamma^{-|\mathbf{s}| - |\boldsymbol{\alpha}|} m^{-|\mathbf{s}| - |\boldsymbol{\alpha}|} \\ &+ (K+1) \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| = K+1}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \int_0^1 (1-y)^K \prod_{k=1}^q (\gamma m + u_k y)^{-s_k - \alpha_k} dy. \end{aligned}$$

It follows that for any  $(\mathbf{u}, \gamma) \in W_0(\mathbf{q}, 1)$  and any  $\mathbf{s} \in \mathcal{D}_{1,q}$ ,

$$\begin{aligned} Z_{1,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma) &= \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| \leq K}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \gamma^{-|\mathbf{s}| - |\boldsymbol{\alpha}|} \zeta(|\mathbf{s}| + |\boldsymbol{\alpha}|) \\ &+ (K+1) \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^q \\ |\boldsymbol{\alpha}| = K+1}} \binom{-\mathbf{s}}{\boldsymbol{\alpha}} \mathbf{u}^{\boldsymbol{\alpha}} \mathcal{R}_K(\mathbf{s}; \mathbf{u}; \gamma; \boldsymbol{\alpha}) \end{aligned}$$

where

$$R_K(\mathbf{s}; \mathbf{u}; \gamma; \boldsymbol{\alpha}) = \sum_{m \geq 1} \int_0^1 (1-y)^K \prod_{k=1}^q (\gamma m + u_k y)^{-s_k - \alpha_k} dy.$$

Let  $\varepsilon > 0$ . We have uniformly in  $m \in \mathbb{N}$ ,  $(\mathbf{u}, \gamma) \in W_\varepsilon(\mathbf{q}, 1)$ ,  $y \in [0, 1]$  and  $k \in \{1, \dots, q\}$ :

$$\begin{aligned} |\gamma m + u_k y| &\geq \Re(\gamma)m + \Re(u_k)y = \varepsilon m + (\Re(\gamma) - \varepsilon)m + \Re(u_k)y \\ &\geq \varepsilon m + (\Re(\gamma) - \varepsilon + \Re(u_k))y \geq \varepsilon m \end{aligned}$$

and

$$|\gamma m + u_k y| \leq |\gamma|m + |u_k| \leq (|\gamma| + |u_k|)m.$$

The theorem of analyticity under the integral sign implies then that

$$(\mathbf{s}, \mathbf{u}, \gamma) \mapsto R_K(\mathbf{s}; \mathbf{u}; \gamma; \boldsymbol{\alpha})$$

is holomorphic in the domain  $\{\mathbf{s} \in \mathbb{C}^q \mid \Re(s_1 + \dots + s_q) > -K\} \times W_\varepsilon(q, 1)$  and the estimate

$$R_K(\mathbf{s}; \mathbf{u}; \gamma; \boldsymbol{\alpha}) \ll_{s, K, \varepsilon} (1 + |\mathbf{u}| + |\gamma|)^{|\mathbf{s}| + K + 1}$$

holds there uniformly in  $(\mathbf{u}, \gamma) \in W_\varepsilon(q, 1)$ . By using in addition the classical properties of Riemann zeta function, we deduce that  $\mathbf{s} \mapsto Z_{1, \mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma)$  has a meromorphic continuation to  $\{\mathbf{s} \in \mathbb{C}^q \mid \Re(s_1 + \dots + s_q) > -K\}$  with poles located in the set  $\mathcal{P}_{1, \mathbf{q}}$  and that the point 2 holds for any  $\boldsymbol{\omega} \in \mathbb{C}^q$  such that  $\Re(\omega_1 + \dots + \omega_q) > -K$ .

By letting  $K \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , we end the proof of Proposition 1 in the case  $n = 1$ .  $\square$

**•Let  $n \in \mathbb{N}$  such that  $n \geq 2$ . We assume that Proposition 1 holds for  $n - 1$ . We will prove that it remains valid for  $n$ :**

Let  $(\mathbf{u}, \gamma) \in W_0(\mathbf{q}, n)$  and  $\mathbf{s} \in \mathcal{D}_{n, \mathbf{q}}$ . Fix  $m_1 \geq 1$  and  $m_2, \dots, m_{n-1} \geq 0$ .

The function  $\varphi(x) := \prod_{i=1}^{q_n} (\gamma_1 m_1 + \dots + \gamma_{n-1} m_{n-1} + \gamma_n x + u_{n,i})^{-s_{n,i}}$  belongs to  $\mathcal{C}^\infty[0, \infty)$  and for all  $k \in \mathbb{N}_0$  and all  $x \in [0, \infty)$ ,

$$\varphi^{(k)}(x) = k! \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^{q_n} \\ |\boldsymbol{\alpha}| = k}} \gamma_n^{|\boldsymbol{\alpha}|} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} \prod_{i=1}^{q_n} (\gamma_1 m_1 + \dots + \gamma_{n-1} m_{n-1} + \gamma_n x + u_{n,i})^{-s_{n,i} - \alpha_i}.$$

Let  $K \in \mathbb{N}_0$ , and let  $\tilde{B}_k$  ( $k \geq 0$ ) be the modified Bernoulli numbers defined by  $\tilde{B}_k := B_k$  for all  $k \neq 1$  and  $\tilde{B}_1 := -B_1 = \frac{1}{2}$ . (In some references  $\tilde{B}_k$  is written as  $B_k$ .)

By applying the Euler-Maclaurin formula to the above  $\varphi(x)$ , we obtain that

$$\begin{aligned} &\sum_{m_n=0}^{\infty} \prod_{i=1}^{q_n} \left( \sum_{j=1}^n \gamma_j m_j + u_{n,i} \right)^{-s_{n,i}} \\ &= \int_0^\infty \prod_{i=1}^{q_n} \left( \sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + u_{n,i} \right)^{-s_{n,i}} dx \\ &\quad + \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^{q_n} \\ |\boldsymbol{\alpha}| \leq K}} \frac{(-1)^{|\boldsymbol{\alpha}|} \tilde{B}_{|\boldsymbol{\alpha}|+1} \gamma_n^{|\boldsymbol{\alpha}|}}{|\boldsymbol{\alpha}|+1} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} \prod_{i=1}^{q_n} \left( \sum_{j=1}^{n-1} \gamma_j m_j + u_{n,i} \right)^{-s_{n,i} - \alpha_i} \end{aligned} \tag{20}$$

$$+(-1)^K \gamma_n^{K+1} \sum_{\substack{\alpha \in \mathbb{N}_0^{q_n} \\ |\alpha|=K+1}} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} \int_0^\infty B_{K+1}(x) \prod_{i=1}^{q_n} \left( \sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + u_{n,i} \right)^{-s_{n,i}-\alpha_i} dx,$$

where  $B_{K+1}(x)$  is the  $(K+1)$ -th periodic Bernoulli polynomial. On the integrand in the last integral of the above, again using Taylor's formula with remainder ([8, (3.4)]), we have

$$\begin{aligned} & \prod_{i=1}^{q_n} \left( \sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + u_{n,i} \right)^{-s_{n,i}} \\ &= \left( \sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x \right)^{-\sum_{i=1}^{q_n} s_{n,i}} \prod_{i=1}^{q_n} \left( 1 + \frac{u_{n,i}}{\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x} \right)^{-s_{n,i}} \quad (21) \\ &= \sum_{\substack{\alpha \in \mathbb{N}_0^{q_n} \\ |\alpha| \leq K}} \left( \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} u_{n,i}^{\alpha_i} \right) \left( \sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x \right)^{-\sum_{i=1}^{q_n} (s_{n,i} + \alpha_i)} \\ &+ (K+1) \sum_{\substack{\alpha \in \mathbb{N}_0^{q_n} \\ |\alpha|=K+1}} \left( \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} u_{n,i}^{\alpha_i} \right) \\ &\quad \times \int_0^1 (1-y)^K \prod_{i=1}^{q_n} \left( \sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + y u_{n,i} \right)^{-s_{n,i}-\alpha_i} dy. \end{aligned}$$

Substituting (20) and (21) into (17) we find that, for any  $K \in \mathbb{N}_0$ , any  $(\mathbf{u}, \gamma) \in W_0(\mathbf{q}, n)$  and any  $\mathbf{s} \in \mathcal{D}_{n, \mathbf{q}}$ :

$$\begin{aligned} Z_{n, \mathbf{q}}(\mathbf{s}; \mathbf{u}; \gamma) &= \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_{n-1} \geq 0}} \frac{1}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (\gamma_1 m_1 + \dots + \gamma_j m_j + u_{j,k})^{s_{j,k}}} \\ &\quad \times \sum_{m_n=0}^{\infty} \frac{1}{\prod_{i=1}^{q_n} (\gamma_1 m_1 + \dots + \gamma_n m_n + u_{n,i})^{s_{n,i}}} \\ &= \sum_{\substack{\alpha \in \mathbb{N}_0^{q_n} \\ |\alpha| \leq K}} \frac{\left( \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} u_{n,i}^{\alpha_i} \right)}{\gamma_n (-1 + \sum_{i=1}^{q_n} (s_{n,i} + \alpha_i))} \\ &\quad \times \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_{n-1} \geq 0}} \frac{1}{\left[ \prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (\sum_{i=1}^j \gamma_i m_i + u_{j,k})^{s_{j,k}} \right] \left( \sum_{j=1}^{n-1} \gamma_j m_j \right)^{\sum_{i=1}^{q_n} (s_{n,i} + \alpha_i) - 1}} \\ &+ \sum_{\substack{\alpha \in \mathbb{N}_0^{q_n} \\ |\alpha| \leq K}} \frac{(-1)^{|\alpha|} \tilde{B}_{|\alpha|+1} \gamma_n^{|\alpha|}}{|\alpha|+1} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_{n-1} \geq 0}} \frac{1}{\left[ \prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (\sum_{i=1}^j \gamma_i m_i + u_{j,k})^{s_{j,k}} \right] (\prod_{i=1}^{q_n} (\sum_{j=1}^{n-1} \gamma_j m_j + u_{n,i})^{s_{n,i} + \alpha_i})} \\
& + \mathcal{R}_{K,n}^1(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) + \mathcal{R}_{K,n}^2(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}), \tag{22}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_{K,n}^1(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) &= (K+1) \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^{q_n} \\ |\boldsymbol{\alpha}| = K+1}} \left( \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} u_{n,i}^{\alpha_i} \right) \\
& \times \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_{n-1} \geq 0}} \frac{\int_0^\infty \int_0^1 (1-y)^K \prod_{i=1}^{q_n} (\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + y u_{n,i})^{-s_{n,i} - \alpha_i} dy dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (\gamma_1 m_1 + \dots + \gamma_j m_j + u_{j,k})^{s_{j,k}}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_{K,n}^2(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) &= (-1)^K \gamma_n^{K+1} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^{q_n} \\ |\boldsymbol{\alpha}| = K+1}} \prod_{i=1}^{q_n} \binom{-s_{n,i}}{\alpha_i} \\
& \times \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_{n-1} \geq 0}} \frac{\int_0^\infty B_{K+1}(x) \prod_{i=1}^{q_n} (\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + u_{n,i})^{-s_{n,i} - \alpha_i} dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (\gamma_1 m_1 + \dots + \gamma_j m_j + u_{j,k})^{s_{j,k}}}.
\end{aligned}$$

The formula (22) is the key for the induction process. In fact, the induction hypothesis implies that the first two terms on the right-hand side of (22) can be continued meromorphically to the whole space, and their poles are located in the set  $\mathcal{P}_{n,\mathbf{q}}$ .

The remaining task is to evaluate  $\mathcal{R}_{K,n}^1(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$  and  $\mathcal{R}_{K,n}^2(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$ . Define

$$\mathcal{D}_{n,\mathbf{q}}(K) := \{ \mathbf{s} \in \mathbb{C}^q \mid \Re \left( \sum_{i=j}^n \sum_{k=1}^{q_i} s_{i,k} \right) > n + 1 - j - K \text{ for all } j = 1, \dots, n \}.$$

Let  $\varepsilon > 0$ . We have uniformly in  $x_1 \geq 1, x_2, \dots, x_n \geq 0, (\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n), y \in [0, 1], j \in \{1, \dots, n\}$  and  $k \in \{1, \dots, q_j\}$ :

$$\begin{aligned}
|\gamma_1 x_1 + \dots + \gamma_j x_j + u_{j,k} y| &\geq \Re(\gamma_1) x_1 + \Re(u_{j,k}) y + \sum_{i=2}^j \Re(\gamma_i) x_i \\
&= \varepsilon x_1 + (\Re(\gamma_1) - \varepsilon) x_1 + \Re(u_{j,k}) y + \sum_{i=2}^j \Re(\gamma_i) x_i \\
&\geq \varepsilon x_1 + (\Re(\gamma_1) - \varepsilon + \Re(u_{j,k})) y + \sum_{i=2}^j \Re(\gamma_i) x_i
\end{aligned}$$



$$\geq \varepsilon x_1 + \sum_{i=2}^j \Re(\gamma_i) x_i \geq \varepsilon \left( \sum_{i=1}^j x_i \right) \quad (23)$$

and

$$\begin{aligned} |\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y| &\leq |\gamma_1| x_1 + \cdots + |\gamma_j| x_j + |u_{j,k} y| \\ &\leq (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|) (x_1 + \cdots + x_j). \end{aligned} \quad (24)$$

Combining (23) and (24) we see that for any  $\varepsilon \in (0, 1)$  and any compact subset  $H$  of  $\mathbb{C}$ , we have uniformly in  $x_1 \geq 1$ ,  $x_2, \dots, x_n \geq 0$ ,  $(\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n)$ ,  $y \in [0, 1]$ ,  $j \in \{1, \dots, n\}$ ,  $k \in \{1, \dots, q_j\}$  and  $s \in H$ :

$$\begin{aligned} &|(\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y)^{-s}| \\ &= |\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y|^{-\Re(s)} e^{\Im(s) \arg(\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y)} \\ &\leq |\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y|^{-\Re(s)} e^{2\pi |\Im(s)|} \\ &\ll_H |\gamma_1 x_1 + \cdots + \gamma_j x_j + u_{j,k} y|^{-\Re(s)} \\ &\ll_H \begin{cases} \varepsilon^{-\Re(s)} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) \geq 0 \\ (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{-\Re(s)} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) < 0 \end{cases} \\ &\ll_{H,\varepsilon} \begin{cases} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) \geq 0 \\ (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|s|} (x_1 + \cdots + x_j)^{-\Re(s)} & \text{if } \Re(s) < 0 \end{cases} \\ &\ll_{H,\varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|s|} (x_1 + \cdots + x_j)^{-\Re(s)}. \end{aligned} \quad (25)$$

We deduce that for any  $K \in \mathbb{N}_0$ , any  $\varepsilon > 0$ , for any  $\boldsymbol{\alpha} \in \mathbb{N}_0^{q_n}$  such that  $|\boldsymbol{\alpha}| = K + 1$ , and any compact subset  $\mathcal{K}$  of  $\mathcal{D}_{n,\mathbf{q}}(K)$ , we have uniformly in  $(\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n)$ , in  $\mathbf{s} \in \mathcal{K}$  and in  $m_1 \geq 1$  and  $m_2, \dots, m_n \geq 0$ :

$$\begin{aligned} &\left| \frac{\int_0^\infty \int_0^1 (1-y)^K \prod_{i=1}^{q_n} (\sum_{j=1}^{n-1} \gamma_j m_j + \gamma_n x + y u_{n,i})^{-s_{n,i} - \alpha_i} dy dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (\gamma_1 m_1 + \cdots + \gamma_j m_j + u_{j,k})^{s_{j,k}}} \right| \\ &\ll_{K,\mathcal{K},\varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}| + |\boldsymbol{\alpha}|} \frac{\int_0^\infty \int_0^1 (1-y)^K (\sum_{j=1}^{n-1} m_j + x)^{-\Re \sum_{i=1}^{q_n} (s_{n,i} + \alpha_i)} dy dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (m_1 + \cdots + m_j)^{\Re(s_{j,k})}} \\ &\ll_{K,\mathcal{K},\varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}| + |\boldsymbol{\alpha}|} \frac{\int_0^\infty (\sum_{j=1}^{n-1} m_j + x)^{-\Re \sum_{i=1}^{q_n} (s_{n,i} + \alpha_i)} dx}{\prod_{j=1}^{n-1} \prod_{k=1}^{q_j} (m_1 + \cdots + m_j)^{\Re(s_{j,k})}} \\ &\ll_{K,\mathcal{K},\varepsilon} \frac{(1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}| + K + 1}}{\left( \prod_{j=1}^{n-1} (m_1 + \cdots + m_j)^{\Re(\sum_{i=1}^{q_j} s_{j,i})} \right) (m_1 + \cdots + m_{n-1})^{\Re(\sum_{i=1}^{q_n} s_{j,i}) + K}}. \end{aligned}$$

In view of (2), the theorem of analyticity under the integral sign implies then that

$$(\mathbf{s}, \mathbf{u}, \boldsymbol{\gamma}) \rightarrow \mathcal{R}_{K,n}^1(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$$

is holomorphic in the domain  $\mathcal{D}_{n,\mathbf{q}}(K) \times W_\varepsilon(\mathbf{q}, n)$  and verifies in it the estimate

$$\mathcal{R}_{K,n}^1(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) \ll_{\mathbf{s}, K, \varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}|+K+1} \text{ uniformly in } (\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n).$$

A similar argument shows that that

$$(\mathbf{s}, \mathbf{u}, \boldsymbol{\gamma}) \rightarrow \mathcal{R}_{K,n}^2(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$$

is holomorphic in the domain  $\mathcal{D}_{n,\mathbf{q}}(K) \times W_\varepsilon(\mathbf{q}, n)$  and the estimate

$$\mathcal{R}_{K,n}^2(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) \ll_{\mathbf{s}, K, \varepsilon} (1 + |\mathbf{u}| + |\boldsymbol{\gamma}|)^{|\mathbf{s}|+K+1}$$

holds there uniformly in  $(\mathbf{u}, \boldsymbol{\gamma}) \in W_\varepsilon(\mathbf{q}, n)$ .

Now we can conclude from (22) that  $\mathbf{s} \mapsto Z_{n,\mathbf{q}}(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$  has the meromorphic continuation to  $\mathcal{D}_{n,\mathbf{q}}(K)$  with poles located in the set  $\mathcal{P}_{n,\mathbf{q}}$  and that the point 2 of Proposition 1 holds for any  $\boldsymbol{\omega} \in \mathcal{D}_{n,\mathbf{q}}(K)$ .

By letting  $K \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , we end the proof of Proposition 1 in the case  $n$ . This finish the proof of Proposition 1.  $\square$

Now we can prove the following necessary result:

**Proposition 2.** *Let  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  be a vectors of complex parameters such that  $\Re(\gamma_j) > 0$  for all  $j = 1, \dots, n$ . Define*

$$V_0(\boldsymbol{\gamma}) := \{\mathbf{u} \in \mathbb{C}^n \mid \Re(u_j + \gamma_j) > 0 \text{ for all } j = 1, \dots, n\}$$

For  $\mathbf{u} \in V_0(\boldsymbol{\gamma})$  and  $\mathbf{s} \in \mathcal{D}_n$ , define

$$Y_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) := \int_{(1,\infty) \times (0,\infty)^{n-1}} \prod_{j=1}^n \left( \sum_{i=1}^j \gamma_i x_i + u_j \right)^{-s_j} dx_n \dots dx_1. \quad (26)$$

Let  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$  such that  $\theta_j + \dots + \theta_n \neq 0$  for all  $j = 1, \dots, n$ . Then, for any  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ , the limit

$$Y_n^\theta(-\mathbf{N}; \mathbf{u}; \boldsymbol{\gamma}) := \lim_{t \rightarrow 0} Y_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma})$$

exists, and we have

$$\begin{aligned} & Y_n^\theta(-\mathbf{N}; \mathbf{u}; \boldsymbol{\gamma}) \\ &= \sum_{I \subset \{1, \dots, n\}} \sum_{\boldsymbol{\alpha} \in \mathcal{J}(I, N)} \frac{(-1)^{n-|I| + \sum_{j \in L(\mathbf{N}, \boldsymbol{\alpha})} (\alpha_j - N_j)} \prod_{j \notin L(\mathbf{N}, \boldsymbol{\alpha})} \binom{N_j}{\alpha_j}}{\prod_{j \in L(\mathbf{N}, \boldsymbol{\alpha})} \alpha_j \binom{\alpha_j - 1}{N_j} \prod_{j \notin I} \left( \sum_{i=j}^n N_i + (n+1-j) - \sum_{i=j}^n \alpha_i \right)} \\ & \quad \times \left( \gamma_1^{|\mathbf{N}| - |\boldsymbol{\alpha}| + n} \prod_{j=1}^n \gamma_j^{-1} \right) \left( \frac{\prod_{j \in L(\mathbf{N}, \boldsymbol{\alpha})} \theta_j}{\prod_{j \in I} (\theta_j + \dots + \theta_n)} \right) \mathbf{u}^\alpha. \end{aligned}$$

**Proof of Proposition 2:**

First we recall from Proposition 1 that  $Y_n(\mathbf{s}; \mathbf{u}; \gamma)$  has a meromorphic continuation to the whole complex space  $\mathbb{C}^n$  and its poles are located in the set

$$\mathcal{P}_n := \bigcup_{j=1}^n \bigcup_{k_j \in \mathbb{N}_0} \{\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n \mid s_j + \dots + s_n = n + 1 - j - k_j\}.$$

Define

$$V_1(\gamma_1) := \{\mathbf{u} \in \mathbb{C}^n \mid |u_j| < \Re(\gamma_1) \text{ for all } j = 1, \dots, n\}. \quad (27)$$

Let  $\mathbf{s} = (s_1, \dots, s_n) \in \mathcal{D}_n$  and we first assume that  $\mathbf{u} \in V_1(\gamma_1)$ . We have uniformly in  $\mathbf{x} = (x_1, \dots, x_n) \in [1, \infty) \times [0, \infty)^{n-1}$ :

$$\left| \frac{u_j}{\sum_{i=1}^j \gamma_i x_i} \right| \leq \frac{|u_j|}{\sum_{i=1}^j \Re(\gamma_i) x_i} \leq \frac{|u_j|}{\Re(\gamma_1)} < 1.$$

Therefore

$$\prod_{j=1}^n \left( \sum_{i=1}^j \gamma_i x_i + u_j \right)^{-s_j} = \sum_{\alpha \in \mathbb{N}_0^n} \binom{-\mathbf{s}}{\alpha} \mathbf{u}^\alpha \prod_{j=1}^n \left( \sum_{i=1}^j \gamma_i x_i \right)^{-s_j - \alpha_j},$$

where the right-hand side is uniform in  $\mathbf{x} = (x_1, \dots, x_n) \in [1, \infty) \times [0, \infty)^{n-1}$ . This implies that for any  $\mathbf{s} = (s_1, \dots, s_n) \in \mathcal{D}_n$ ,

$$Y_n(\mathbf{s}; \mathbf{u}; \gamma) = \sum_{\alpha \in \mathbb{N}_0^n} \binom{-\mathbf{s}}{\alpha} \mathbf{u}^\alpha Y_n(\mathbf{s} + \alpha; \gamma),$$

where  $Y_n(\mathbf{s}; \gamma)$  is defined by (13). Applying Lemma 1 we obtain that for any  $\mathbf{s} = (s_1, \dots, s_n) \in \mathcal{D}_n$ ,

$$Y_n(\mathbf{s}; \mathbf{u}; \gamma) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\gamma_1^{-|\mathbf{s}| - |\alpha| + n} \binom{-\mathbf{s}}{\alpha} \mathbf{u}^\alpha}{(\gamma_1 \dots \gamma_n) \prod_{j=1}^n (s_j + \dots + s_n + \alpha_j + \dots + \alpha_n + j - n - 1)}. \quad (28)$$

Moreover, since  $\mathbf{u} \in V_1(\gamma_1)$ , the right-hand side of (28) is uniformly convergent in any compact subset of  $\mathbb{C}^n \setminus \mathcal{P}_n$ . It follows that the meromorphic continuation of  $Y_n(\mathbf{s}; \mathbf{u}; \gamma)$  is given by (28) for any  $\mathbf{s} \in \mathbb{C}^n \setminus \mathcal{P}_n$ .

Let  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$  such that  $\sum_{i=j}^n \theta_i \neq 0$  for all  $j$  and  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$ . Set  $\delta := \frac{1}{2} \min \{(1 + |\theta_j|)^{-1}, |\theta_j + \dots + \theta_n|^{-1}; j = 1, \dots, n\} \in (0, 1/2)$ . From (28) we obtain that

$$Y_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \gamma) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\gamma_1^{|\mathbf{N}| + n - |\alpha| - t|\boldsymbol{\theta}|} \mathbf{u}^\alpha}{(\gamma_1 \dots \gamma_n) \prod_{j=1}^n (\theta_j + \dots + \theta_n)} G_{\mathbf{N}, \alpha, \boldsymbol{\theta}}(t)$$

for any  $t \in U_\delta \setminus \{0\}$ , where  $G_{\mathbf{N}, \alpha, \theta}(t)$  is defined by (14). By using point 2 of Lemma 3, it follows from Lebesgue's dominated convergence theorem that  $Y_n^\theta(-\mathbf{N}; \mathbf{u}; \gamma) := \lim_{t \rightarrow 0} Y_n(-\mathbf{N} + t\theta; \mathbf{u}; \gamma)$  exists and that

$$Y_n^\theta(-\mathbf{N}; \mathbf{u}; \gamma) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{\gamma_1^{|\mathbf{N}|+n-|\alpha|} \mathbf{u}^\alpha}{(\gamma_1 \cdots \gamma_n) \prod_{j=1}^n (\theta_j + \cdots + \theta_n)} G_{\mathbf{N}, \alpha, \theta}(0), \quad (29)$$

where  $G_{\mathbf{N}, \alpha, \theta}(0)$  is defined in Lemma 3. Moreover, points 3 and 4(b) of Lemma 3 imply that  $G_{\mathbf{N}, \alpha, \theta}(0) = 0$  if  $\alpha \notin \{0, |\mathbf{N}| + n\}^n$ . It follows that the sum in the right-hand side of (29) is finite.

Therefore by using the expression of  $G_{\mathbf{N}, \alpha, \theta}(0)$  given by Lemma 3 and by arranging the terms we obtain that

$$\begin{aligned} Y_n^\theta(-\mathbf{N}; \mathbf{u}; \gamma) &= \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |K(\mathbf{N}, \alpha)| = |L(\mathbf{N}, \alpha)|}} \frac{\gamma_1^{|\mathbf{N}|+n-|\alpha|} \mathbf{u}^\alpha}{(\gamma_1 \cdots \gamma_n) \prod_{j=1}^n (\theta_j + \cdots + \theta_n)} \\ &\quad \times \frac{(-1)^{n-|K(\mathbf{N}, \alpha)|} \left( \prod_{j \in L(\mathbf{N}, \alpha)} \frac{(-1)^{\alpha_j - N_j} \theta_j}{\alpha_j \binom{\alpha_j - 1}{N_j}} \right) \prod_{j \notin L(\mathbf{N}, \alpha)} \binom{N_j}{\alpha_j}}{\prod_{j \notin K(\mathbf{N}, \alpha)} \left( \frac{(N_j + \cdots + N_n) + (n+1-j) - (\alpha_j + \cdots + \alpha_n)}{\theta_j + \cdots + \theta_n} \right)} \\ &= \sum_{I \subset \{1, \dots, n\}} \sum_{\alpha \in \mathcal{J}(I, \mathbf{N})} \frac{\gamma_1^{|\mathbf{N}|+n-|\alpha|} \mathbf{u}^\alpha}{(\gamma_1 \cdots \gamma_n) \prod_{j=1}^n (\theta_j + \cdots + \theta_n)} \\ &\quad \times \frac{(-1)^{n-|I|} \left( \prod_{j \in L(\mathbf{N}, \alpha)} \frac{(-1)^{\alpha_j - N_j} \theta_j}{\alpha_j \binom{\alpha_j - 1}{N_j}} \right) \prod_{j \notin L(\mathbf{N}, \alpha)} \binom{N_j}{\alpha_j}}{\prod_{j \notin I} \left( \frac{(N_j + \cdots + N_n) + (n+1-j) - (\alpha_j + \cdots + \alpha_n)}{\theta_j + \cdots + \theta_n} \right)}. \end{aligned} \quad (30)$$

Fix  $\mathbf{N} \in \mathbb{N}_0^n$ . We now extend the region of  $\mathbf{u}$  for which the proposition holds. Denote the last member of (30) by  $\psi(\mathbf{u})$ . Then clearly  $\psi(\mathbf{u})$  is defined and analytic on the set  $V_0(\gamma)$ . Moreover, Corollary 2 implies that for any  $\mathbf{u} \in V_0(\gamma)$

$$Y_n^\theta(-\mathbf{N} + t\theta; \mathbf{u}; \gamma) = \sum_{k=0}^n \frac{y_{-k}(\mathbf{u}; -\mathbf{N}, \theta, \gamma)}{t^k} + O(t) \text{ as } t \rightarrow 0,$$

where for any  $k = 0, \dots, n$ ,  $\mathbf{u} \mapsto y_{-k}(\mathbf{u}; -\mathbf{N}, \theta, \gamma)$  is analytic in the domain  $V_0(\gamma)$ . On the other hand, (30) implies that for any  $\mathbf{u} \in V_1(\gamma_1)$ ,

$$y_0(\mathbf{u}; -\mathbf{N}, \theta, \gamma) = \psi(\mathbf{u}) \text{ and } y_{-k}(\mathbf{u}; -\mathbf{N}, \theta, \gamma) = 0 \text{ for all } k = 1, \dots, n. \quad (31)$$

Since  $V_1(\gamma_1)$  is a non-empty open subset of the convex (and hence connected) open set  $V_0(\gamma)$ , it follows then from identity theorem for holomorphic functions that (31) holds for any  $\mathbf{u} \in V_0(\gamma)$ .

This ends the proof of Proposition 2.  $\square$

## 2.3 An extension of Raabe's lemma

Define for any  $\delta \in \mathbb{R}$ ,

$$\mathcal{H}_n(\delta) := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \mid \Re(z_i) > \delta \text{ for all } i = 1, \dots, n\}.$$

**Lemma 4. (An extension of Raabe's lemma)**

Let  $\delta > 0$ . Let  $g : \mathcal{H}_n(-\delta) \rightarrow \mathbb{C}$  be an analytic function in  $\mathcal{H}_n(-\delta)$  such that there exists two constants  $K > 0$  and  $c \in (0, \pi)$  such that

$$|g(\mathbf{z})| \leq K e^{c(|z_1| + \dots + |z_n|)} \quad \forall \mathbf{z} \in \mathcal{H}_n(-\delta). \quad (32)$$

Define for any  $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{H}_n(-\delta)$ ,

$$f(\mathbf{x}) = \int_{[0,1]^n} g(\mathbf{x} + \mathbf{y}) \, d\mathbf{y}. \quad (33)$$

Assume that  $f$  is a polynomial of degree at most  $d$ . Then,  $g$  is also a polynomial of degree at most  $d$ . Moreover, if we write  $f(\mathbf{x}) = \sum_{\alpha} a_{\alpha} \mathbf{x}^{\alpha} = \sum_{\alpha} a_{\alpha} \prod_{i=1}^n x_i^{\alpha_i}$  (with  $\alpha = (\alpha_1, \dots, \alpha_n)$ ), then

$$g(\mathbf{x}) = \sum_{\alpha} a_{\alpha} \prod_{i=1}^n B_{\alpha_i}(x_i), \quad (34)$$

where the  $B_k(x)$  are the classical Bernoulli polynomials.

**Remark:** Raabe's transform (33) is an important operator which makes it possible to derive several properties of a Dirichlet series from its associated Dirichlet integral. For the history of Raabe's formula, see E. Friedman and S. Ruijsenaars [10, p.367]. E. Friedman and A. Pereira [9] proved this lemma under the assumption that both  $f$  and  $g$  are polynomials. For our aim in the present paper, we only assume in Lemma 4 that  $g$  is an analytic function in a suitable domain satisfying the estimate (32) which is necessary for Carlson's theorem.

**Proof of Lemma 4:**

We will proceed by induction on  $n$ :

• **The case  $n = 1$ :**

The theorem of differentiation under the integral sign implies that for any  $x \in \mathcal{H}_1(-\delta)$ ,

$$0 = f^{(d+1)}(x) = \int_{[0,1]} g^{(d+1)}(x+y) \, dy = g^{(d)}(x+1) - g^{(d)}(x).$$

It follows that

$$g^{(d)}(k) = g^{(d)}(0) \quad \text{for all } k \in \mathbb{N}_0.$$

Let  $z \in \mathbb{C}$  such that  $\Re(z) \geq 0$ . The Cauchy formula and (32) imply that

$$|g^{(d)}(z)| = \left| \frac{d!}{2\pi i} \int_{|t-z|=\delta/2} \frac{g(t)}{(t-z)^{d+1}} dt \right| \leq K' e^{c|z|},$$

where  $K' = K d! \left(\frac{\delta}{2}\right)^{-d} e^{c\delta/2} > 0$ .

Then it follows then from Carlson's classical theorem (F. Carlson [3]; see 5.81 in page 186 of [20]) that

$$g^{(d)}(z) = g^{(d)}(0) \quad \text{for all } z \in \mathbb{C} \text{ such that } \Re(z) \geq 0.$$

Thus,  $g$  is a polynomial of degree at most  $d$ .

Now since we know that  $f$  and  $g$  are both polynomials, (34) is a consequence of the Lemma of Friedman and Pereira (see Lemma 2.4 of [9]) of Raabe type. This ends the proof of Lemma 4 in the case  $n = 1$ .  $\square$

• **Let  $n \in \mathbb{N}$ . Assume that Lemma 4 is true for function in  $n-1$  variables, we will prove that it remains valid for function in  $n$  variables:**

Let  $\delta > 0$  and let  $g : \mathcal{H}_n(-\delta) \rightarrow \mathbb{C}$  be an analytic function satisfying the assumptions of Lemma 4. Let  $\beta \in \mathbb{N}_0^n$  such that  $|\beta| > d$ . The Cauchy formula and (32) imply that there exists two  $K' > 0$  and  $c \in (0, \pi)$  such that

$$|\partial^\beta g(\mathbf{z})| \leq K' e^{c(|z_1| + \dots + |z_n|)} \quad \text{for all } \mathbf{z} \in \mathcal{H}_n(-\delta/2). \quad (35)$$

Fix  $\mathbf{z}' = (z_1, \dots, z_{n-1}) \in \mathcal{H}_{n-1}(-\delta/2)$  and define  $h : \mathcal{H}_1(-\delta/2) \rightarrow \mathbb{C}$  by

$$h(z_n) := \int_{[0,1]^{n-1}} \partial^\beta g(z_1 + a_1, \dots, z_{n-1} + a_{n-1}, z_n) da_1 \dots da_{n-1}.$$

It is easy to see that  $h$  is analytic in  $\mathcal{H}_1(-\delta/2)$  and that (35) implies that

$$|h(z_n)| \leq K'(\mathbf{z}') e^{c|z_n|} \quad \text{for all } z_n \in \mathcal{H}_1(-\delta/2),$$

where  $K'(\mathbf{z}') = K' \left(\frac{e^c - 1}{e}\right)^{n-1} e^{c(|z_1| + \dots + |z_{n-1}|)} > 0$ .

On the other hand, we have for any  $z_n \in \mathcal{H}_1(-\delta/2)$ ,

$$\begin{aligned} \int_{[0,1]} h(z_n + a_n) da_n &= \int_{[0,1]^n} \partial^\beta g(z_1 + a_1, \dots, z_n + a_n) da_1 \dots da_n \\ &= \partial^\beta \left( \int_{[0,1]^n} g(z_1 + a_1, \dots, z_n + a_n) da_1 \dots da_n \right) \\ &= \partial^\beta f(\mathbf{z}) = 0 \end{aligned}$$

because  $|\beta| > d$ . The case  $n = 1$  implies then that for any  $z_n \in \mathcal{H}_1(-\delta/2)$ ,  $h(z_n) = 0$ . As a conclusion we proved that

$$\int_{[0,1]^{n-1}} \partial^\beta g(z_1 + a_1, \dots, z_{n-1} + a_{n-1}, z_n) da_1 \dots da_{n-1} = 0 \quad (36)$$

for all  $\mathbf{z} = (z_1, \dots, z_n) \in \mathcal{H}_n(-\delta/2)$ .

Now Fix  $z_n \in \mathcal{H}_1(-\delta/2)$  and define  $\ell : \mathcal{H}_{n-1}(-\delta/2) \rightarrow \mathbb{C}$  by

$$\ell(z_1, \dots, z_{n-1}) = \partial^\beta g(z_1, \dots, z_{n-1}, z_n).$$

It is easy to see that  $\ell$  is analytic in  $\mathcal{H}_{n-1}(-\delta/2)$  and that (35) imply that

$$|\ell(\mathbf{z}')| \leq K''(z_n) e^{c(|z_1| + \dots + |z_{n-1}|)} \quad \text{for all } \mathbf{z}' = (z_1, \dots, z_{n-1}) \in \mathcal{H}_{n-1}(-\delta/2),$$

where  $K''(z_n) = K' e^{c|z_n|} > 0$ .

It follows then from our induction hypothesis and (36) that

$$\ell(z_1, \dots, z_{n-1}) = 0 \quad \text{for all } \mathbf{z}' = (z_1, \dots, z_{n-1}) \in \mathcal{H}_{n-1}(-\delta/2)$$

and hence that for any  $\beta \in \mathbb{N}_0^n$  with  $|\beta| > d$  we have

$$\partial^\beta g(z_1, \dots, z_{n-1}, z_n) = 0 \quad \text{for all } \mathbf{z} = (z_1, \dots, z_n) \in \mathcal{H}_n(-\delta/2).$$

It follows that  $g$  is a polynomial of degree at most  $d$ . Now since we know that both  $f$  and  $g$  are polynomials, (34) is again a consequence of Raabe's Lemma of Friedman and Pereira. This ends the induction argument and the proof of Lemma 4.  $\square$

We end this section with the following useful lemma. This lemma is maybe not new. But we give a proof of it in order to be self-contained. For  $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$ , define

$$\mathcal{H}_n(\delta) := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \mid \Re(z_j) > \delta_j \text{ for all } j = 1, \dots, n\}.$$

**Lemma 5.** *Let  $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$  and  $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$  such that  $\mu_j \geq \delta_j$  for all  $j = 1, \dots, n$ . Let  $f : \mathcal{H}_n(\delta) \rightarrow \mathbb{C}$  be an analytic function. Assume that  $f(\mathbf{x}) = 0$  for all  $\mathbf{x} \in \prod_{j=1}^n (\mu_j, \infty)$ . Then  $f(\mathbf{z}) = 0$  for all  $\mathbf{z} \in \mathcal{H}_n(\delta)$ .*

### Proof of Lemma 5:

We will prove the lemma by induction on  $n$ .

If  $n = 1$  the lemma follows from the classical principle of permanence. Let  $n \geq 2$ . Assume that Lemma 5 is true for functions of  $n - 1$  variables. We will prove that it remains true for functions of  $n$  variables.

Fix  $x_1, \dots, x_{n-1} \in \mathbb{R}$  such that  $x_i > \mu_i$  for all  $i = 1, \dots, n - 1$ . Define the function  $F : \mathcal{H}_1(\mu_n) \rightarrow \mathbb{C}$  by  $F(z) = f(x_1, \dots, x_{n-1}, z)$ . It follows from our assumptions that  $F$  is analytic in the domain  $\mathcal{H}_1(\mu_n)$  and that  $F(x) = 0$  for all  $x \in (\mu_n, \infty)$ . We deduce then from principle of permanence that  $F(z) = 0$  for all  $z \in \mathcal{H}_1(\mu_n)$ . That is, we have

$$f(x_1, \dots, x_{n-1}, z_n) = 0 \text{ for all } (x_1, \dots, x_{n-1}, z_n) \in \left( \prod_{j=1}^{n-1} (\mu_j, \infty) \right) \times \mathcal{H}_1(\mu_n). \quad (37)$$

Now fix  $z_n \in \mathbb{C}$  such that  $\Re(z_n) > \mu_n$ . Let  $\boldsymbol{\mu}' = (\mu_1, \dots, \mu_{n-1})$  and define  $g : \mathcal{H}_{n-1}(\boldsymbol{\mu}') \rightarrow \mathbb{C}$  by  $g(z_1, \dots, z_{n-1}) = f(z_1, \dots, z_{n-1}, z_n)$ . Then  $g$  is analytic in  $\mathcal{H}_{n-1}(\boldsymbol{\mu}')$  and (37) implies that  $g(x_1, \dots, x_{n-1}) = 0$  for all  $(x_1, \dots, x_{n-1}) \in \prod_{j=1}^{n-1}(\mu_j, \infty)$ . The induction hypothesis implies then that

$$g(z_1, \dots, z_{n-1}) = 0 \quad \text{for all } (z_1, \dots, z_{n-1}) \in \mathcal{H}_{n-1}(\boldsymbol{\mu}').$$

We deduce that

$$f(z_1, \dots, z_n) = 0 \quad \text{for all } \mathbf{z} \in \mathcal{H}_n(\boldsymbol{\mu}).$$

This ends the proof of Lemma 5 since  $\mathcal{H}_n(\boldsymbol{\mu})$  is a non-empty open subset of the domain  $\mathcal{H}_n(\boldsymbol{\delta})$ .  $\square$

## 2.4 Completion of the proof of Theorem 1

Fix  $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}_0^n$  and  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{C}^n$ . Assume that

$$\theta_j + \dots + \theta_n \neq 0 \quad \text{for all } j = 1, \dots, n.$$

Set

$$W := \{(\mathbf{u}, \boldsymbol{\gamma}) \in \mathbb{C}^n \times \mathbb{C}^n \mid \Re(\gamma_j) > 0 \text{ and } \Re(u_j + \gamma_1) > 0 \text{ for all } j = 1, \dots, n\}.$$

For  $(\mathbf{u}, \boldsymbol{\gamma}) \in W$  and  $\mathbf{s} \in \mathcal{D}_n$ , we consider  $Y_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma})$  defined by (26) and

$$Z_n(\mathbf{s}; \mathbf{u}; \boldsymbol{\gamma}) := \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{1}{\prod_{j=1}^n (\gamma_1 m_1 + \dots + \gamma_j m_j + u_j)^{s_j}}. \quad (38)$$

We know from Corollary 2 that for any  $(\mathbf{u}, \boldsymbol{\gamma}) \in W$ , the functions

$$t \mapsto Y_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) \text{ and } t \mapsto Z_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma})$$

are meromorphic and have at most a pole of order  $n$  at  $t = 0$ . Write

$$Y_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{k=0}^n \frac{y_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma})}{t^k} + O(t) \quad \text{as } t \rightarrow 0,$$

and

$$Z_n(-\mathbf{N} + t\boldsymbol{\theta}; \mathbf{u}; \boldsymbol{\gamma}) = \sum_{k=0}^n \frac{z_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma})}{t^k} + O(t) \quad \text{as } t \rightarrow 0.$$

Corollary 2 implies then that for any  $k = 0, \dots, n$ , the functions

$$(\mathbf{u}, \boldsymbol{\gamma}) \mapsto y_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \text{ and } (\mathbf{u}, \boldsymbol{\gamma}) \mapsto z_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \quad (39)$$



are analytic in the domain  $W$  and

$$y_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}) = \int_{[0,1]^n} z_{-k}(\mathbf{u}(\mathbf{b}); -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}) d\mathbf{b} \quad (40)$$

holds in that domain, where  $\mathbf{u}(\mathbf{b}) = (u_1(\mathbf{b}), \dots, u_n(\mathbf{b})) \in \mathbb{C}^n$  with  $\mathbf{b} = (b_1, \dots, b_n)$ ,  $u_j(\mathbf{b}) = u_j + \sum_{i=1}^j \gamma_i b_i$  for all  $j$ .

For any  $\boldsymbol{\gamma} \in \mathbb{C}^n$  such that  $\Re(\gamma_j) > 0$  for all  $j = 1, \dots, n$ , define

$$\mathcal{V}(\boldsymbol{\gamma}) := \left\{ \mathbf{u} \in \mathbb{C}^n \mid \Re(u_j + \gamma_1) > \Re\left(\sum_{i=1}^j \gamma_i\right) + 1 \text{ for all } j = 1, \dots, n \right\}.$$

Temporarily we assume that  $\boldsymbol{\gamma} \in (1, \infty)^n$  and  $\mathbf{u} \in \mathcal{V}(\boldsymbol{\gamma})$ . It is easy to see that for all  $\mathbf{a} \in \mathcal{H}_n(-1)$ , and all  $j = 1, \dots, n$ ,

$$\Re(\mathbf{u}_j(\mathbf{a}) + \gamma_1) = \Re(u_j + \gamma_1) + \sum_{i=1}^j \gamma_i \Re(a_i) > \Re(u_j + \gamma_1) - \sum_{i=1}^j \gamma_i > 1,$$

that is for all  $\mathbf{a} \in \mathcal{H}_n(-1)$ ,

$$\mathbf{u}(\mathbf{a}) \in V_1(\boldsymbol{\gamma}) = \{\mathbf{z} \in \mathbb{C}^n \mid \Re(z_j + \gamma_1) > 1 \text{ for all } j = 1, \dots, n\}.$$

Define for any  $k = 0, \dots, n$  and for any  $\mathbf{a} \in \mathcal{H}_n(-1)$ ,

$$f_k(\mathbf{a}) := y_{-k}(\mathbf{u}(\mathbf{a}); -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \quad \text{and} \quad g_k(\mathbf{a}) := z_{-k}(\mathbf{u}(\mathbf{a}); -\mathbf{N}, \boldsymbol{\theta}; \boldsymbol{\gamma}).$$

Corollary 2 implies then that for any  $k = 0, \dots, n$  the following three points hold:

1.  $f_k$  and  $g_k$  are analytic functions in  $\mathcal{H}_n(-1)$ ;
2.  $f_k(\mathbf{x}) = \int_{[0,1]^n} g_k(\mathbf{x} + \mathbf{y}) d\mathbf{y}$  for all  $\mathbf{x} \in \mathcal{H}_n(-1)$ ;
3. there exists a constant  $M = M(\mathbf{N}, \boldsymbol{\theta}) > 0$  such that, uniformly in  $\mathbf{a} \in \mathcal{H}_n(-1)$ , we have  $g_k(\mathbf{a}) \ll_{\mathbf{N}, \boldsymbol{\theta}, \mathbf{u}, \boldsymbol{\gamma}} (1 + |\mathbf{a}|)^M$ .

On the other hand, Proposition 2 implies that for all  $\mathbf{a} \in \mathcal{H}_n(-1)$ ,

$$f_k(\mathbf{a}) = 0 \quad \text{for all } k = 1, \dots, n$$

and

$$f_0(\mathbf{a}) = \sum_{I \subset \{1, \dots, n\}} \sum_{\boldsymbol{\alpha} \in \mathcal{J}(I, N)} A(\mathbf{N}, \boldsymbol{\alpha}, \boldsymbol{\theta}; \boldsymbol{\gamma}) \prod_{j=1}^n \left( u_j + \sum_{i=1}^j \gamma_i a_i \right)^{\alpha_j},$$

where

$$A(\mathbf{N}, \boldsymbol{\alpha}, \boldsymbol{\theta}; \gamma) = \frac{(-1)^{n-|I|+\sum_{j \in L(\mathbf{N}, \boldsymbol{\alpha})}(\alpha_j - N_j)} \prod_{j \notin L(\mathbf{N}, \boldsymbol{\alpha})} \binom{N_j}{\alpha_j}}{\prod_{j \in L(\mathbf{N}, \boldsymbol{\alpha})} \alpha_j \binom{\alpha_j - 1}{N_j} \prod_{j \notin I} \left( \sum_{i=j}^n N_i + (n+1-j) - \sum_{i=j}^n \alpha_i \right)} \\ \times \left( \gamma_1^{|\mathbf{N}| - |\boldsymbol{\alpha}| + n} \prod_{j=1}^n \gamma_j^{-1} \right) \left( \frac{\prod_{j \in L(\mathbf{N}, \boldsymbol{\alpha})} \theta_j}{\prod_{j \in I} (\theta_j + \dots + \theta_n)} \right).$$

In particular,  $f_0(\mathbf{a})$  is a polynomial in  $\mathbf{a}$ .

We deduce then from Lemma 4 that for all  $\mathbf{a} \in \mathcal{H}_n(-1)$ ,

$$g_k(\mathbf{a}) = 0 \quad \text{for all } k = 1, \dots, n$$

and

$$g_0(\mathbf{a}) = \sum_{I \subset \{1, \dots, n\}} \sum_{\boldsymbol{\alpha} \in \mathcal{J}(I, N)} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n, \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} \tilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) A(\mathbf{N}, \boldsymbol{\alpha}, \boldsymbol{\theta}; \gamma) \prod_{j=1}^n B_{k_j}(a_j),$$

where the polynomials  $\tilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k})$  are defined by

$$\prod_{j=1}^n \left( \sum_{i=1}^j \gamma_i X_i + u_j \right)^{\alpha_j} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n, \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} \tilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) \mathbf{X}^{\mathbf{k}} = \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n, \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} \tilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) X_1^{k_1} \dots X_n^{k_n}. \quad (41)$$

By taking  $\mathbf{a} = 0$ , we obtain that, for all  $\gamma \in (1, \infty)^n$  and all  $\mathbf{u} \in \mathcal{V}(\gamma)$ :

$$z_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \gamma) = 0 \quad \text{for all } k = 1, \dots, n \quad (42)$$

and

$$z_0(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \gamma) = \sum_{I \subset \{1, \dots, n\}} \sum_{\boldsymbol{\alpha} \in \mathcal{J}(I, N)} \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n, \\ |\mathbf{k}| \leq |\boldsymbol{\alpha}|}} \tilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) A(\mathbf{N}, \boldsymbol{\alpha}, \boldsymbol{\theta}; \gamma) \prod_{j=1}^n B_{k_j}. \quad (43)$$

Since  $\tilde{c}_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) = c_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k}) \gamma_1^{k_1} \dots \gamma_n^{k_n}$  (where  $c_n(\mathbf{u}; \boldsymbol{\alpha}, \mathbf{k})$  is defined by (8)), the right-hand side of (43) coincides with the right-hand side of (9).

Moreover, for any fixed  $\gamma \in (1, \infty)^n$ ,  $\mathcal{V}(\gamma)$  is a non-empty open subset of the domain  $V_0(\gamma)$  and we know that for all  $k = 0, \dots, n$ ,  $\mathbf{u} \mapsto z_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \gamma)$  is analytic in  $V_0(\gamma)$ . It follows then by analytic continuation that for any  $\gamma \in (1, \infty)^n$  the identities (42) and (43) hold for any  $\mathbf{u}$  in the whole domain  $V_0(\gamma)$ .

Now fix  $\mathbf{u} \in \mathbb{C}^n$  and set  $\eta(\mathbf{u}) := \max \{0, -\Re(u_1), \dots, -\Re(u_n)\}$ . Define

$$\mathcal{G}(\mathbf{u}) := \{\gamma \in \mathbb{C}^n \mid \Re(\gamma_1) > \eta(\mathbf{u}) \text{ and } \Re(\gamma_j) > 0 \text{ for all } j = 2, \dots, n\}.$$

From the definition of  $V_0(\gamma)$  and  $W$ , it is easy to see that  $\{\mathbf{u}\} \times \mathcal{G}(\mathbf{u}) \subset W$ . It follows then from (39) that

$$\gamma \mapsto z_{-k}(\mathbf{u}; -\mathbf{N}, \boldsymbol{\theta}; \gamma) \text{ is analytic in the domain } \mathcal{G}(\mathbf{u}).$$

We already know from the above that the identities (42) and (43) hold for  $\gamma \in (1, \infty)^n \cap \mathcal{G}(\mathbf{u})$ . Lemma 5 implies then that for any  $\mathbf{u} \in \mathbb{C}^n$  the identities (42) and (43) hold for any  $\gamma$  in the whole domain  $\mathcal{G}(\mathbf{u})$ . This ends the proof of Theorem 1.  $\square$

### 3 Proof of Theorem 2

Now we proceed to the second topic of the present paper.

Fix  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  such that  $\Re(\gamma_j) > 0$  and  $\Re(b_j) > -\Re(\gamma_1)$  and  $\frac{b_n - b_{n-1}}{\gamma_n} \notin -\mathbb{N}_0$  for all  $j = 1, \dots, n$ . Fix also  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_{n-1}) \in (\mathbb{T} \setminus \{1\})^{n-1}$ .

The zeta function

$$\zeta_{n,n-1}(\mathbf{s}, \gamma, \mathbf{b}, \boldsymbol{\mu}_{n-1}) = \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{\prod_{j=1}^{n-1} \mu_j^{m_j}}{\prod_{j=1}^n (\gamma_1 m_1 + \dots + \gamma_j m_j + b_j)^{s_j}}$$

is absolutely convergent (see [14]) in the region  $\mathcal{D}_n$ , hence especially in its subregion

$$\mathcal{A}_n = \{\mathbf{s} \in \mathbb{C}^n \mid \Re s_j > 1 \ (1 \leq j \leq n)\}.$$

Recall the Mellin-Barnes integral formula:

$$(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z dz, \quad (44)$$

where  $s, \lambda \in \mathbb{C}$ ,  $\Re s > 0$ ,  $\lambda \neq 0$ ,  $|\arg \lambda| < \pi$ ,  $-\Re s < c < 0$ , and the path of the integral is the vertical line  $\Re z = c$ .

Assume  $n \geq 2$ . Further assume temporarily that  $\mathbf{s} \in \mathcal{A}_n$  and

$$\Re(b_n - b_{n-1}) > 0. \quad (45)$$

Our starting point is the decomposition

$$\begin{aligned} & (\gamma_1 m_1 + \dots + \gamma_n m_n + b_n)^{-s_n} \\ &= (\gamma_1 m_1 + \dots + \gamma_{n-1} m_{n-1} + b_{n-1})^{-s_n} \\ & \times \left( 1 + \frac{\gamma_n m_n + b_n - b_{n-1}}{\gamma_1 m_1 + \dots + \gamma_{n-1} m_{n-1} + b_{n-1}} \right)^{-s_n}. \end{aligned} \quad (46)$$

Under the assumption (45) we see that

$$\left| \arg \left( \frac{\gamma_n m_n + b_n - b_{n-1}}{\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1}} \right) \right| < \pi,$$

hence the above decomposition (46) is valid, and using (44) we obtain

$$\begin{aligned} & (\gamma_1 m_1 + \cdots + \gamma_n m_n + b_n)^{-s_n} \\ &= (\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1})^{-s_n} \\ & \times \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z) \Gamma(-z)}{\Gamma(s_n)} \left( \frac{\gamma_n m_n + b_n - b_{n-1}}{\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1}} \right)^z dz, \end{aligned} \quad (47)$$

where  $-\Re s_n < c < 0$ . But since  $\mathbf{s} \in \mathcal{A}_n$ , we have  $\Re s_n > 1$ , so we may assume (more strongly)

$$-\Re s_n < c < -1. \quad (48)$$

Substituting (47) into (10) (with  $k = n - 1$ ) and changing the order of integration and summation, we have

$$\begin{aligned} & \zeta_{n,n-1}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-1}) \\ &= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z) \Gamma(-z)}{\Gamma(s_n)} \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{\prod_{l=1}^{n-1} \mu_l^{m_l}}{\prod_{j=1}^{n-2} (\gamma_1 m_1 + \cdots + \gamma_j m_j + b_j)^{s_j}} \\ & \times (\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1})^{-s_{n-1} - s_n - z} (\gamma_n m_n + b_n - b_{n-1})^z dz \\ &= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z) \Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(z), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}) \\ & \times \zeta \left( -z, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^z dz, \end{aligned} \quad (49)$$

where  $\mathbf{s}_{n-1}^*(z) = (s_1, \dots, s_{n-2}, s_{n-1} + s_n + z)$ . (Under the assumption (48), both of the above two zeta factors in the integrand are convergent.)

Let  $N$  be a positive integer, and now we shift the path of integration to  $\Re z = N + 1/2$ . This is possible because two gamma-factors on the numerator decay very rapidly when  $|\Im z|$  becomes large. The relevant poles are  $z = -1$  (from the Hurwitz zeta factor) and  $z = 0, 1, 2, \dots, N$  (from  $\Gamma(-z)$ ). Counting the residues, we obtain

$$\begin{aligned} & \zeta_{n,n-1}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-1}) \\ &= \frac{1}{s_n - 1} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(-1), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\ & + \sum_{l=0}^N \binom{-s_n}{l} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(l), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \end{aligned} \quad (50)$$

$$\begin{aligned}
& \times \zeta \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l \\
& + \frac{1}{2\pi i} \int_{(N+1/2)} \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1, n-1}(\mathbf{s}_{n-1}^*(z), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\
& \times \zeta \left( -z, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^z dz.
\end{aligned}$$

Since  $\zeta_{n-1, n-1}$  is entire, the poles (in  $z$ ) of the integrand of the above integral are  $z = -1, 0, 1, 2, \dots$  and  $z = -s_n, -s_n - 1, -s_n - 2, \dots$ . Therefore the above integral can be continued holomorphically to the region satisfying  $\Re(-s_n) < N + 1/2$ , that is,

$$\{\mathbf{s} \in \mathbb{C}^n \mid \Re s_n > -N - 1/2\}.$$

Since  $N$  is arbitrary, we can show from (50) that  $\zeta_{n, n-1}(\mathbf{s}_n, \gamma_n, \mathbf{b}_n, \boldsymbol{\mu}_{n-1})$  can be continued meromorphically to the whole space  $\mathbb{C}^n$ . Moreover, again noting that  $\zeta_{n-1, n-1}$  is entire, we find that the only possible singularity is the hyperplane  $s_n = 1$ .

Furthermore, here we can weaken the restriction (45). As we mentioned in Section 1, The Hurwitz zeta-function  $\zeta(s, \alpha)$  can be defined for any complex  $\alpha$  except for the case when  $\alpha = -l$ ,  $l \in \mathbb{N}_0$ . Therefore, continue (50) with respect to  $\mathbf{b}_n$ , we can say that (50) is valid for any  $\mathbf{b}_n$  satisfying

$$\Re b_j > -\Re \gamma_1 \quad (1 \leq j \leq n) \quad \text{and} \quad \frac{b_n - b_{n-1}}{\gamma_n} \notin -\mathbb{N}_0. \quad (51)$$

Let  $-\mathbf{N} = (-N_1, \dots, -N_n)$ , where  $N_j$  ( $1 \leq j \leq n$ ) are non-negative integers. Then  $\mathbf{s} = -\mathbf{N}$  is a regular point of the function  $\zeta_{n, n-1}(\mathbf{s}, \gamma_n, \mathbf{b}_n, \boldsymbol{\mu}_{n-1})$ .

Put  $\mathbf{s} = -\mathbf{N}$  on (50). Then the integral is equal to 0, because of the factor  $\Gamma(s_n) = \Gamma(-N_n)$  on the denominator. Also, when  $l > N_n$ , then the binomial coefficient  $\binom{N_n}{l}$  is equal to 0. Therefore we obtain the following explicit formula:

$$\begin{aligned}
& \zeta_{n, n-1}(-\mathbf{N}, \gamma, \mathbf{b}, \boldsymbol{\mu}_{n-1}) \\
& = -\frac{1}{N_n + 1} \zeta_{n-1, n-1}(-\mathbf{N}_{n-1}^*(1), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\
& + \sum_{l=0}^{N_n} \binom{N_n}{l} \zeta_{n-1, n-1}(-\mathbf{N}_{n-1}^*(-l), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\
& \quad \times \zeta \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l
\end{aligned} \quad (52)$$

The special values  $\zeta_{n-1, n-1}(-\mathbf{k}_{n-1}, \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1})$  (where  $\mathbf{k}_{n-1} = (k_1, \dots, k_{n-1}) \in \mathbb{N}_0^{n-1}$ ) are evaluated explicitly by de Crisenoy [4] in terms of special values of the Lerch zeta-function  $\zeta_{\mu_j}(s)$ . Since

$$-N_{n-1} - N_n + l \leq -N_{n-1} \leq 0$$

for  $l \leq N_n$ , we can apply the result of de Crisenoy (Theorem B of [4]) to the factors  $\zeta_{n-1,n-1}$  appearing on the right-hand side of the above.

Let  $b'_1 = b_1$ ,  $b'_j = b_j - (\gamma_2 + \cdots + \gamma_j)$  ( $2 \leq j \leq n-1$ ). Then we can write

$$\zeta_{n-1,n-1}(\mathbf{s}_{n-1}, \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) = \sum_{m_1, \dots, m_{n-1} \geq 1} \frac{\prod_{j=1}^{n-1} \mu_j^{m_j}}{\prod_{j=1}^{n-1} (\gamma_1 m_1 + \cdots + \gamma_j m_j + b'_j)^{s_j}},$$

which agrees with de Crisenoy's notation. His Theorem B then implies

$$\begin{aligned} & \zeta_{n-1,n-1}(-\mathbb{N}_{n-1}^*(l), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\ &= \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{n-1} \\ |\mathbf{k}| \leq |\mathbb{N}_{n-1}^*(l)|}} \tilde{c}_{n-1}(\mathbf{b}'_{n-1}; \mathbb{N}_{n-1}^*(l), \mathbf{k}) \prod_{j=1}^{n-1} \gamma_j^{k_j} \zeta_{\mu_j}(-k_j), \end{aligned}$$

where  $\tilde{c}_{n-1}(\mathbf{b}'_{n-1}; \mathbb{N}_{n-1}^*(l), \mathbf{k})$  is that defined by (41). Applying this to the right-hand side of (52), we obtain the assertion of Theorem 2.  $\square$

**Remark:** For  $\boldsymbol{\mu} \in (\mathbb{T} \setminus \{1\})^n$ , we can apply the same argument as above to  $\zeta_{n,n}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu})$ . The result is that the special values  $\zeta_{n,n}(-\mathbf{N}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu})$  can be written in terms of  $\zeta_{n-1,n-1}(-\mathbb{N}_{n-1}^*(-l), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1})$  and special values of

$$\phi\left(s, \frac{b_n - b_{n-1}}{\gamma_n}, \mu_n\right) = \sum_{m=0}^{\infty} \mu_n^m \left(m + \frac{b_n - b_{n-1}}{\gamma_n}\right)^{-s}.$$

## 4 Proof of Theorem 3

### 4.1 Behavior of $\zeta_{n,n-1}$ around its singularity $s_n = 1$

As a preparation, here we consider the behavior of  $\zeta_{n,n-1}$  around its singularity  $s_n = 1$ . We will use in the sequel of this section the notations of Section 3.

Let  $s_n = 1 + \delta_n$ , where  $\delta_n$  is a small (non-zero) complex number. Then

$$\begin{aligned} & \frac{1}{s_n - 1} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(-1), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\ &= \frac{1}{\delta_n} \zeta_{n-1,n-1}((s_1, \dots, s_{n-2}, s_{n-1} + \delta_n), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\ &= \frac{1}{\delta_n} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}, \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\ &+ \frac{\partial}{\partial s_{n-1}} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}, \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} + O(|\delta_n|), \end{aligned}$$

so from (50) we have

$$\begin{aligned}
& \zeta_{n,n-1}((s_1, \dots, s_{n-1}, 1 + \delta_n), \gamma_n, \mathbf{b}_n, \boldsymbol{\mu}_{n-1}) \\
&= \frac{1}{\delta_n} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}, \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\
&+ \frac{\partial}{\partial s_{n-1}} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}, \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\
&+ \sum_{l=0}^N \binom{-1}{l} \zeta_{n-1,n-1}((s_1, \dots, s_{n-2}, s_{n-1} + 1 + l), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\
&\quad \times \zeta\left(-l, \frac{b_n - b_{n-1}}{\gamma_n}\right) \gamma_n^l \\
&+ \frac{1}{2\pi i} \int_{(N+1/2)} \Gamma(1+z) \Gamma(-z) \zeta_{n-1,n-1}((s_1, \dots, s_{n-2}, s_{n-1} + 1 + z), \\
&\quad \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \zeta\left(-z, \frac{b_n - b_{n-1}}{\gamma_n}\right) \gamma_n^z dz + O(|\delta_n|) \\
&= \frac{1}{\delta_n} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}, \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \gamma_n^{-1} \\
&+ B(\mathbf{s}_{n-1}, \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) + O(|\delta_n|),
\end{aligned} \tag{53}$$

say. So far we have worked under the assumption  $n \geq 2$ . However when  $n = 1$ , we see that

$$\begin{aligned}
\zeta_{1,0}(1 + \delta_1, \gamma_1, \mathbf{b}_1, \boldsymbol{\mu}_0) &= \sum_{m_1=1}^{\infty} (\gamma_1 m_1 + b_1)^{-1-\delta_1} \\
&= \gamma_1^{-1-\delta_1} \zeta(1 + \delta_1, b_1/\gamma_1) - b_1^{-1-\delta_1} = \frac{1}{\delta_1} \gamma_1^{-1} + (\text{constant}) + O(|\delta_1|),
\end{aligned}$$

so (53) is valid also for  $n = 1$  with the convention  $\zeta_{0,0} = 1$ .

## 4.2 Proof of Theorem 3

Now consider the case  $k = n - 2$  ( $n \geq 2$ ). Fix  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$  such that  $\Re(\gamma_j) > 0$  and  $\Re(b_j) > -\Re(\gamma_1)$  and  $\frac{b_n - b_{n-1}}{\gamma_n} \notin -\mathbb{N}_0$  for all  $j = 1, \dots, n$ . Fix also  $\boldsymbol{\mu}_{n-2} = (\mu_1, \dots, \mu_{n-2}) \in (\mathbb{T} \setminus \{1\})^{n-2}$ .

Assume  $\mathbf{s} \in \mathcal{A}_n$ . Analogous to (49), this time we obtain

$$\begin{aligned}
& \zeta_{n,n-2}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-2}) \\
&= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z) \Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-2}(\mathbf{s}_{n-1}^*(z), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2})
\end{aligned} \tag{54}$$

$$\times \zeta \left( -z, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^z dz,$$

where  $-\Re s_n < c < -1$ . This time the factor

$$\zeta_{n-1,n-2}(\mathbf{s}_{n-1}^*(z), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2})$$

is not entire, but its pole  $s_{n-1} + s_n + z = 1$ , that is,  $z = 1 - s_{n-1} - s_n$  is irrelevant when we shift the path from  $\Re z = c$  to  $\Re z = N + 1/2$ , because  $\Re(1 - s_{n-1} - s_n) < -\Re s_n < c$ . Therefore, analogous to (50), we have

$$\begin{aligned} & \zeta_{n,n-2}(\mathbf{s}, \gamma, \mathbf{b}, \boldsymbol{\mu}_{n-2}) \\ &= \frac{1}{s_n - 1} \zeta_{n-1,n-2}(\mathbf{s}_{n-1}^*(-1), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \gamma_n^{-1} \\ &+ \sum_{l=0}^N \binom{-s_n}{l} \zeta_{n-1,n-2}(\mathbf{s}_{n-1}^*(l), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \\ &\quad \times \zeta \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l \\ &+ \frac{1}{2\pi i} \int_{(N+1/2)} \frac{\Gamma(s_n + z) \Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-2}(\mathbf{s}_{n-1}^*(z), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \\ &\quad \times \zeta \left( -z, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^z dz. \end{aligned} \tag{55}$$

Here, the (unique) singularity of  $\zeta_{n-1,n-2}(\mathbf{s}_{n-1}^*(l), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2})$  is  $\mathbf{s}_{n-1}^*(l) = 1$ , that is,  $s_{n-1} + s_n = 1 - l$  ( $l = -1, 0, 1, 2, \dots, N$ ). Letting  $N \rightarrow \infty$  we obtain the meromorphic continuation of  $\zeta_{n,n-2}(\mathbf{s}, \gamma, \mathbf{b}, \boldsymbol{\mu}_{n-2})$ , and its (possible) singularities are

$$\begin{cases} s_n = 1, \\ s_{n-1} + s_n = 2, 1, 0, -1, -2, \dots \end{cases} \tag{56}$$

Now we want to evaluate the value of  $\zeta_{n,n-2}(\mathbf{s}, \gamma, \mathbf{b}, \boldsymbol{\mu}_{n-2})$  at  $\mathbf{s} = -\mathbf{N} \in -\mathbb{N}_0^n$ . The above (56) shows that  $\mathbf{s} = -\mathbf{N}$  is on a singular locus.

Let  $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$ , where  $\delta_j$ s are small (non-zero) complex numbers, and observe the right-hand side of (55) with  $\mathbf{s} = -\mathbf{N} + \boldsymbol{\delta}$ . Since  $-\mathbf{N}_{n-1}^*(l) = -N_{n-1} - N_n + l$ , the only relevant singularity of  $\zeta_{n-1,n-2}$  factor appears when  $l = N_{n-1} + N_n + 1$ . Analogous to (52), we have

$$\begin{aligned} & \zeta_{n,n-2}(-\mathbf{N} + \boldsymbol{\delta}, \gamma, \mathbf{b}, \boldsymbol{\mu}_{n-2}) \\ &= -\frac{1}{N_n + 1} \zeta_{n-1,n-2}(-\mathbf{N}_{n-1}^*(-1), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \gamma_n^{-1} \\ &+ \sum_{l=0}^{N_n} \binom{N_n}{l} \zeta_{n-1,n-2}(-\mathbf{N}_{n-1}^*(l), \gamma_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \end{aligned} \tag{57}$$



$$\times \zeta \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l + R + O \left( \max_{1 \leq j \leq n} |\delta_j| \right),$$

where  $R$  denotes the contribution coming from the term  $l = N_{n-1} + N_n + 1$ . Using (53), we can evaluate  $R$  as follows:

$$\begin{aligned} R &= \binom{N_n - \delta_n}{N_{n-1} + N_n + 1} \\ &\times \zeta_{n-1, n-2}((-N_1 + \delta_1, \dots, -N_{n-2} + \delta_{n-2}, 1 + \delta_{n-1} + \delta_n), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-2}) \\ &\times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^{N_{n-1} + N_n + 1} \\ &= \frac{(N_n - \delta_n)(N_n - 1 - \delta_n) \cdots (-\delta_n) \cdots (-N_{n-1} - \delta_n)}{(N_{n-1} + N_n + 1)!} \\ &\times \left\{ \frac{1}{\delta_{n-1} + \delta_n} \zeta_{n-2, n-2}(-\mathbf{N}_{n-2} + \boldsymbol{\delta}_{n-2}, \boldsymbol{\gamma}_{n-2}, \mathbf{b}_{n-2}, \boldsymbol{\mu}_{n-2}) \gamma_{n-1}^{-1} \right. \\ &\quad \left. + B(-\mathbf{N}_{n-2} + \boldsymbol{\delta}_{n-2}, \boldsymbol{\gamma}_{n-2}, \mathbf{b}_{n-2}, \boldsymbol{\mu}_{n-2}) + O(|\delta_{n-1} + \delta_n|) \right\} \\ &\times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^{N_{n-1} + N_n + 1} \\ &= \frac{(N_n - \delta_n)(N_n - 1 - \delta_n) \cdots (-\delta_n) \cdots (-N_{n-1} - \delta_n)}{(N_{n-1} + N_n + 1)! (\delta_{n-1} + \delta_n)} \\ &\times \zeta_{n-2, n-2}(-\mathbf{N}_{n-2} + \boldsymbol{\delta}_{n-2}, \boldsymbol{\gamma}_{n-2}, \mathbf{b}_{n-2}, \boldsymbol{\mu}_{n-2}) \\ &\times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_{n-1}^{-1} \gamma_n^{N_{n-1} + N_n + 1} \\ &\quad + O(|\delta_n|), \end{aligned} \tag{58}$$

which describes the situation of indeterminacy. We may understand the behavior of  $\zeta_{n, n-2}$  around the point  $\mathbf{s} = -\mathbf{N}$  from (57) and (58). This ends the proof of Theorem 3.  $\square$

## 5 The power sum case

By the method in the present paper, it is possible to discuss multiple zeta-functions of more general form (12), whose denominators are not linear forms. The general theory will be postponed to [7], but we conclude this paper with the discussion of the case of the following special type of non-linear forms.

Let  $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{N}^n$ , and consider

$$\zeta_{n,k}(\mathbf{s}, \mathbf{h}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k) = \sum_{\substack{m_1 \geq 1 \\ m_2, \dots, m_n \geq 0}} \frac{\prod_{j=1}^k \mu_j^{m_j}}{\prod_{j=1}^n (\gamma_j m_j^{h_j} + b_j)^{s_j}}. \quad (59)$$

This is obviously a generalization of  $\zeta_{n,k}(\mathbf{s}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_k)$ , and can be treated in a quite similar way as in the linear case in Theorems 2 and 3. The analogue of (49) is

$$\begin{aligned} & \zeta_{n,n-1}(\mathbf{s}, \mathbf{h}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-1}) \\ &= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(z), \mathbf{h}_{n-1}, \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\ & \quad \times \zeta\left(-z, h_n, \frac{b_n - b_{n-1}}{\gamma_n}\right) \gamma_n^z dz, \end{aligned} \quad (60)$$

where  $-\Re s_n < c < -1$  and

$$\zeta(s, h, b) = \sum_{m=0}^{\infty} \frac{1}{(m^h + b)^s} \quad (h \in \mathbb{N}, b \in \mathbb{C}, |\arg b| < \pi). \quad (61)$$

The analytic properties of  $\zeta(s, h, b)$  can also be studied by using the Mellin-Barnes formula. In fact,

$$\begin{aligned} \zeta(s, h, b) &= b^{-s} + \sum_{m=1}^{\infty} m^{-hs} (1 + b/m^h)^{-s} \\ &= b^{-s} + \frac{1}{2\pi i} \sum_{m=1}^{\infty} m^{-hs} \int_{(c_1)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \left(\frac{b}{m^h}\right)^z dz \end{aligned} \quad (62)$$

( $-\Re s < c_1 < 0$ ), which is, after changing the order of integration and summation,

$$= b^{-s} + \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \zeta(h(s+z)) b^z dz. \quad (63)$$

To assure the convergence of  $\zeta(h(s+z))$ , we have to choose  $c_1$  satisfying  $h^{-1} - \Re s < c_1 < 0$ . Now, shift the path to  $\Re z = N + 1/2$ , and count the residues of relevant poles at  $z = 0, 1, 2, \dots, N$ . We obtain

$$\begin{aligned} \zeta(s, h, b) &= b^{-s} + \sum_{l=0}^N \binom{-s}{l} \zeta(h(s+l)) b^l \\ & \quad + \frac{1}{2\pi i} \int_{(N+1/2)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \zeta(h(s+z)) b^z dz. \end{aligned} \quad (64)$$

Considering the situation  $N \rightarrow \infty$ , we find that (64) gives the meromorphic continuation of  $\zeta(s, h, b)$  to the whole plane. The Riemann zeta factor on the right-hand side gives the poles (of order at most 1) at  $s = -l + h^{-1}$  ( $l \in \mathbb{N}_0$ ).

When  $h = 1$ , the poles  $s = -l + h^{-1} = -l + 1$  are cancelled with the binomial factor for  $l \geq 1$ , so the only pole is  $s = 1$ . This is of course the case of the Hurwitz zeta-function. When  $h \geq 2$ , all of  $s = -l + h^{-1}$  ( $l \in \mathbb{N}_0$ ) are really poles. The residue at  $s = -l + h^{-1}$  is

$$\frac{1}{h} \binom{l - h^{-1}}{l} b^l. \quad (65)$$

Now let us go back to (60), and shift the path to  $\Re z = N + 1/2$ . This time, the relevant poles are  $z = 0, 1, 2, \dots, N$  (from  $\Gamma(-z)$ ) and  $z = -1$  (if  $h_n = 1$ ) or  $z = l - h_n^{-1}$  ( $0 \leq l \leq N$ , if  $h_n \geq 2$ ). Analogous to (50), we obtain

$$\begin{aligned} & \zeta_{n,n-1}(\mathbf{s}, \mathbf{h}, \boldsymbol{\gamma}, \mathbf{b}, \boldsymbol{\mu}_{n-1}) \quad (66) \\ &= \sum_{l=0}^N \delta_1(l, h_n) \frac{\Gamma(s_n + l - h_n^{-1}) \Gamma(-l + h_n^{-1})}{\Gamma(s_n)} \\ & \quad \times \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(l - h_n^{-1}), \mathbf{h}_{n-1}, \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\ & \quad \times \frac{1}{h_n} \binom{l - h_n^{-1}}{l} \left( \frac{b_n - b_{n-1}}{\gamma_n} \right)^l \gamma_n^{-l + h_n^{-1}} \\ &+ \sum_{l=0}^N \binom{-s_n}{l} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(l), \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\ & \quad \times \zeta \left( -l, h_n, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l \\ &+ \frac{1}{2\pi i} \int_{(N+1/2)} \frac{\Gamma(s_n + z) \Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-1}(\mathbf{s}_{n-1}^*(z), \mathbf{h}_n, \boldsymbol{\gamma}_{n-1}, \mathbf{b}_{n-1}, \boldsymbol{\mu}_{n-1}) \\ & \quad \times \zeta \left( -z, h_n, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^z dz, \end{aligned}$$

where

$$\delta_1(l, h_n) = \begin{cases} 1 & \text{if } h_n \geq 2 \text{ or } h_n = 1, l = 0 \\ 0 & \text{if } h_n = 1, l \geq 1. \end{cases}$$

Put  $\mathbf{s} = -\mathbf{N}$ . When  $h_n = 1$ , only the term corresponding to  $l = 0$  remains in the first sum on the right-hand side of the above, the situation is almost the same as (52). While when  $h_n \geq 2$ , then the first sum vanishes because of the factor  $\Gamma(s_n)$  in the denominator. Therefore this case can also be handled as in the situation of (52).

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